# THE PRINCIPLES
# OF
# PROJECTIVE GEOMETRY

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# THE PRINCIPLES
# OF
# PROJECTIVE GEOMETRY
# APPLIED TO THE
# STRAIGHT LINE AND CONIC

by

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# PREFACE

THIS book has been written in the hope of placing in the hands of the pupil, who has mastered the portions of Euclid usually read, most, if not all, of the Pure Geometry which he requires in order to proceed to an Honours degree in Mathematics at any one of the Universities of Oxford, Cambridge, London and Manchester. While the subject has been primarily considered from the projective point of view, considerable trouble has been taken to deduce the more important metrical properties of Conics from the projective theorems with which they are related. In an addendum a certain number of elementary theorems of a non-projective nature have been collected for reference.

The author trusts that this book may do something to encourage the student not to neglect the methods of pure geometry. In every other branch of mathematics analysis now reigns supreme—even in geometry it is fast gaining a predominance. Twenty years' experience as a Teacher of Projective Geometry and ten years' experience as an examiner of the University of London have led the author to regard this as a misfortune. When the great landmarks of Projective Geometry—the theorems of Pascal and Brianchon, of Carnot and Desargues, together with their immediate consequences—are clearly placed before the student, the author has found that even in the younger student an enthusiasm is aroused which is wanting in his study of other branches of Mathematics. In the examination room it has been found that students who have mastered and absorbed the principles of Pure Geometry have taken a superior place to those who depend on a facility for handling analytical expressions. Such a facility with practice may undoubtedly be acquired by most pupils, but for all who are worthy to take a mathematical degree the study of Pure Geometry is a matter of primary importance.

To the works of Salmon and Cremona the author—like all others who now attempt to deal with the subject—is deeply indebted. He has freely consulted the examination papers of the Universities of Oxford, Cambridge and London, which set the standard of geometrical teaching in this country. To the Savilian Professor of Geometry in the University of Oxford he has to express his indebtedness for much valuable assistance and advice. He has freely consulted the notes of the lectures delivered by the late Professor H. S. Smith. To Mr G. S. Le Beau, Mr S. G. Soal and other members of the mathematical staff of the East London College he offers his sincere thanks for their assistance more especially in connexion with the proof sheets of this book.

The hopelessness of tracing the various theorems to their original discoverers has led the author to abstain from a task which is beyond his power to perform satisfactorily, and as a consequence names have only been associated with theorems when they are in ordinary use for their identification. In most cases the enunciations of theorems have been given before the proofs and an effort has been made to present the proofs in a form suitable for reproduction. In the case of most of the important theorems alternative proofs have been given. The properties of the involution have been largely used as they render it possible to overcome some of the difficulties which otherwise arise through points and lines being real in certain cases and in certain cases imaginary. The last chapter is to some extent an introduction to the future consideration of the "imaginary" in connexion with the conic. This subject has been postponed for a subsequent volume in which the author hopes to deal with it from a geometrical point of view.

J. L. S. H.

East London College,
*March* 1913.

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# NOTE

The student, who has no previous knowledge of the subject, may with advantage read as a first course the large type portions of Chapters I to VIII, X, XII, XIII and XIV. It is also advisable in most cases for the student to attempt the examples for himself before referring to the solutions given in the text. The word "line" is used throughout the book to signify a straight line although out of respect to custom the term "straight line" has been frequently retained. The author would be pleased to be informed of any mistakes or misprints.

# CHAPTER I

## NOTATION. PRINCIPLE OF DUALITY. DEFINITIONS

### 1. Notation.

Geometrical figures and the proofs of theorems are rendered more intelligible, when a uniform notation is employed. In this book, the following notation is adopted:

Points are denoted by the capital letters $A, B, C, \ldots\ldots X, Y, Z$.
Lines „ „ „ small letters $a, b, c, \ldots\ldots x, y, z$.
Planes „ „ „ Greek letters $\alpha, \beta, \gamma, \ldots\ldots \chi, \psi, \omega$.

The point where the line $a$ meets the plane $\alpha$ may be denoted by $a\alpha$; if the lines $a$ and $b$ are situated in the same plane their point of intersection may be similarly denoted by $ab$; the line of intersection of the planes $\alpha$ and $\beta$ may be denoted by $\alpha\beta$; the line joining the points $A$ and $B$ by $AB$, while $Aa$ may be taken to represent the plane through the point $A$ and the line $a$.

Hence it is seen that

Points may be denoted by $a\alpha$, $b\beta$, $c\gamma$; $\alpha\beta\gamma$;
or if $a$ $b$ $c$ are coplanar by $ab$, $bc$, $ca$.
Lines „ „ „ „ $\alpha\beta, \beta\gamma, \ldots$; $AB, BC, \ldots$ .
Planes „ „ „ „ $Aa, Bb, \ldots$; $ABC, \ldots$ .

This method of representation may be carried still further. Thus a meaning may be attached to such expressions as $\alpha . BC$; $A . \beta\gamma$; $\alpha . Bc$; $A . \beta c$. The first represents the point of intersection of the line $BC$ with the plane $\alpha$; the second the plane through the point $A$ and the line $\beta\gamma$; the third the line of intersection of the planes $\alpha$ and $Bc$, and the last the line joining the point $A$ to the point $\beta c$.

It should be noticed in the last case that if $Ac . \beta$ had been written in place of $A . \beta c$ the meaning would not have been the same. $Ac . \beta$ represents the line of intersection of the planes $Ac$ and $\beta$ which is not the same as the line joining the point $A$ to the point $\beta c$.

In cases where there is no fear of confusion angles are denoted by Greek letters.

## 2. Principle of Duality.

Every figure in a plane may be regarded as being constructed either of points or of lines. If it is regarded as made up primarily of points, the lines which occur in it are looked upon as the connectors of pairs of points. If it is considered to be constructed by means of lines, the points in it are determined as the intersections of pairs of lines. In the former case, the curves, which occur in the figure, may be looked upon as consisting of series of points and the tangent at any point is the line joining two consecutive points on the curve. The curve is in this case termed the *locus* of the points situated on it. In the latter case, curves are determined by an infinite number of straight lines which touch them and the point of intersection of any two consecutive tangents gives a point on the curve. The curve is then called the *envelope* of the lines which touch it. Hence, if a figure be constructed or a theorem be proved by using points as the fundamental elements, another figure may be constructed or another theorem be proved in identically the same way by substituting lines for points; thus in geometry theorems occur in pairs. This is called the *Principle of Duality.*

It is usual to prove a theorem in the first instance looking upon points as the fundamental elements and to deduce the corresponding theorem—which is termed the *correlative*—by substituting lines for points. The full importance of this principle will be gradually realised together with the method of its application. As an illustration certain definitions and theorems—which will be proved hereafter—together with their correlatives, are given in the next article.

## 3. Instances of the Principle of Duality.

| | |
|---|---|
| Two points $A$ and $B$ determine a straight line $AB$. | Two straight lines $a$ and $b$ determine a point $ab$. |
| The line $AB$ is the locus of a series of points which are situated on the line. | The point $ab$ is the envelope of the group of lines which pass through the point. |
| Any series of points, situated on the same straight line, is called *a range of points.* | Any group of lines, which pass through the same point, is called *a pencil of rays.* |
| Any four points $A$, $B$, $C$, $D$—no three of which are collinear—determine a *complete quadrangle.* | Any four straight lines $a$, $b$, $c$, $d$—no three of which are concurrent—determine a *complete quadrilateral.* |

In the figure

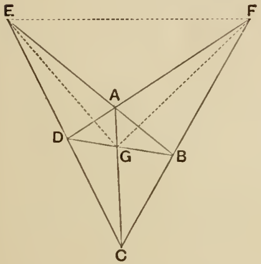
the lines $AB$ and $DC$ determine $E$,
„ „ $BC$ and $DA$ „ $F$,
„ „ $BD$ and $CA$ „ $G$.

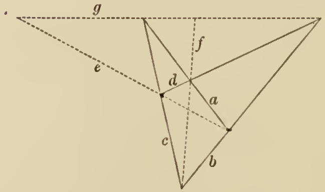
The points $E$, $F$, $G$ are termed the diagonal points of the quadrangle and the triangle $EFG$ the *diagonal points triangle* of the quadrangle.

In the figure

the points $ab$ and $dc$ determine $e$,
„ „ $bc$ and $da$ „ $f$,
„ „ $bd$ and $ca$ „ $g$.

The lines $e, f, g$ are termed the diagonals of the quadrilateral and the triangle $efg$ the *diagonal triangle* of the quadrilateral.





The following are instances of theorems and their correlatives which will be proved in the following chapters.

(*a*) If two triangles $ABC$ and $A'B'C'$ are such that the straight lines $AA'$, $BB'$, $CC'$ intersect at a point $S$, then the three points of intersection of corresponding sides, viz.:

$$AB \, . \, A'B',$$
$$BC \, . \, B'C',$$
$$CA \, . \, C'A'$$

lie on a straight line $s$. (Art. 13 *a*.)

(*b*) If two complete quadrangles $ABCD$, $A'B'C'D'$ are such that five pairs of sides, $AB$ and $A'B'$, $BC$ and $B'C'$, $CA$ and $C'A'$, $AD$ and $A'D'$, $BD$ and $B'D'$ intersect in five points on a straight

If two triangles $abc$ and $a'b'c'$ are such that the points $aa'$, $bb'$, $cc'$ lie on a straight line $s$, then the lines joining the corresponding vertices, viz.:

$$ab \, . \, a'b',$$
$$bc \, . \, b'c',$$
$$ca \, . \, c'a'$$

pass through a point $S$. (Art. 13 *a*.)

If two complete quadrilaterals $abcd$, $a'b'c'd'$ are such that five pairs of vertices, $ab$ and $a'b'$, $bc$ and $b'c'$; $ca$ and $c'a'$, $ad$ and $a'd'$, $bd$ and $b'd'$, lie on five straight lines through a point $S$, then the line $s$, then the remaining pair $CD$ and $C'D'$ also intersect on the line $s$. (Art. 13 *b*.)

(*c*) If the three sides of a variable triangle pass through three collinear points $A$, $A'$ and $S$, and two of the vertices move on fixed lines $s$ and $i$, then the locus of the third vertex is a straight line which passes through the point $si$. (Art. 71 (1).)

remaining pair $cd$ and $c'd'$ also lie on a straight line through $S$. (Art. 13 *b*.)

If the three vertices of a variable triangle are situated on three concurrent straight lines $a$, $a'$ and $s$, and two of the sides pass through fixed points $S$ and $I$, then the third side passes through a fixed point which lies on the line $SI$. (Art. 71 (1).)

In the above (*c*) is a restatement of (*a*).

### 4. Definitions.

The following definitions will be seen to be closely connected with the principle of duality as already explained.

(1) A *Range* or *Row* of points $A$, $B$, $C$, ... is a figure composed of points lying on a straight line termed the *base*.

(2) A *flat pencil*, or shortly, a *pencil* of lines $a$, $b$, $c$, ... is a figure composed of straight lines lying in the same plane and radiating from a point termed the *centre* or *vertex*.

(3) A *plane figure* (plane of points or plane of lines) is a figure which consists of points and straight lines, such that the points of intersection of the lines and the lines joining the points lie in a *plane*.

### 5. Extension of the Principle of Duality.

The Principle of Duality may also be applied to figures in space. In all there are three forms of the principle of duality including the case already considered.

(1) *Figures in a plane* in which the point and straight line are the fundamental elements, the duality being between the point and line.

(2) *Figures in the sheaf* (sheaf of lines, sheaf of planes) which are made up of lines and planes which all pass through a given point. The duality in this case is between the line and the plane. If a sphere be described with its centre at the given point, the lines and planes determine points and great circles, on the sphere. Hence the consideration of such a sheaf leads to the geometry of figures on a sphere and to the results obtained by spherical trigonometry.

(3) *A solid figure* (figure of planes, figure of points) which consists of planes and points. In this figure straight lines are determined either as the intersection of planes or as the connectors of points and the duality is between planes and points.

In the consideration of the preceding *axial pencils* arise, which consist of planes, which all pass through a given line termed the axis of the pencil. Thus, by analogy, it is possible to speak of a *line of points*, *line of planes*, as a figure made up of points which lie on a given line and of the planes which pass through it.

### 6. Connexion between the Principle of Duality and Polar Reciprocation.

The student, who is acquainted with polar reciprocation, will notice that in this case, as in that of the principle of duality, one figure is deduced from the other by substituting line for point and point for line. The methods of proof in the two cases are, however, different. In the Principle of Duality the same proof is interpreted in two different ways to prove a theorem and its correlative. In Polar Reciprocation a theorem is proved and another theorem is deduced therefrom by means of the properties of pole and polar. It was, however, by means of Polar Reciprocation that the Principle of Duality was first arrived at and its importance realised.

# CHAPTER II

## MEASUREMENT OF DISTANCES AND ANGLES. PROJECTIVE FORMS ANHARMONIC

### 7. Measurement of Distances.

The expression $AB$ is used in geometry with two different meanings. According to our definition (Art. 1) it represents the straight line joining the points $A$ and $B$, and carries with it no idea of length. It is, however, also used to denote the distance measured along this line from $A$ to $B$. It is sometimes an advantage to denote this distance by the symbol $\overline{AB}$. This will be done in some cases, but in accordance with general custom $AB$ will be written for $\overline{AB}$ when there is no fear of ambiguity.

As a convention it is usual to assume that $\overline{BA} = -\overline{AB}$ so that

$$\overline{AB} + \overline{BA} = 0.$$

Hence if $A$, $B$, $C$ be on the same straight line

$$\overline{AB} + \overline{BC} + \overline{CA} = 0,$$

whatever be their relative positions. This formula is of importance because in a slightly altered form it enables all distances measured along a straight line to be expressed in terms of the distances of various points on the line from some given point, which given point may be looked upon either as an origin or as a variable point on the line. The formula then becomes

$$\overline{AB} = \overline{OB} - \overline{OA} = \overline{PB} - \overline{PA},$$

$O$ being considered as an origin and $P$ as a variable point on the line.

*To prove that if $A$, $B$, $C$, $D$ are any four collinear points*

$$\overline{BC} \, . \, \overline{AD} + \overline{CA} \, . \, \overline{BD} + \overline{AB} \, . \, \overline{CD} = 0.$$

If all the lengths are expressed as distances from $A$, this relation becomes

$$(\overline{AC} - \overline{AB}) \, \overline{AD} + \overline{CA} \, (\overline{AD} - \overline{AB}) + \overline{AB} \, (\overline{AD} - \overline{AC}) = 0,$$

which is identically true.

*To prove that if $A$, $B$, $C$, $O$ are any four collinear points*

$$\overline{OA}^2 . \overline{BC} + \overline{OB}^2 . \overline{CA} + \overline{OC}^2 . \overline{AB} = -\overline{BC} . \overline{CA} . \overline{AB}.$$

Express all lengths in terms of distances from $O$, then

$$\begin{aligned} -\overline{BC} . \overline{CA} . \overline{AB} &= -(OC - OB)(OA - OC)(OB - OA) \\ &= OA^2(OC - OB) + OB^2(OA - OC) + OC^2(OB - OA) \\ &\qquad + OA . OB . OC - OA . OB . OC \\ &= OA^2(OC - OB) + OB^2(OA - OC) + OC^2(OB - OA) \\ &= OA^2 . BC + OB^2 . CA + OC^2 . AB. \end{aligned}$$

*Examples*: (1) *If $O$, $A$, $B$ are collinear prove that*

$$OA^2 + OB^2 = AB^2 + 2 . OA . OB.$$

(2) *If $C$ be the middle point of $AB$ prove that*

(*a*) $OC = \frac{1}{2}(OA + OB)$,

(*b*) $OA . OB = OC^2 - AC^2$,

(*c*) $OA^2 + OB^2 = CA^2 + CB^2 + 2 . OC^2$,

(*d*) $OA^2 - OB^2 = 2 . AB . CO$.

(3) *Show that the second of the relations proved in the text holds if $O$ is not on the straight line $ABC$.*

The relations connecting the distances of points situated on a straight line may be written in either of two ways according as they are regarded as relations, generally symmetrical, connecting the distances between the points, or as relations connecting the distances of certain variable points, $P$, $Q$, ... from certain fixed points, $A$, $B$, $C$, ... .

The more important are as follows, viz.:

*Three points.* (i) $AB + BC + CA = 0$ or $AB = PB - PA$ ........................(*a*).

*Four points.* (ii) $AB . CD + CA . BD + BC . AD = 0$

or $PA . BC + PB . CA + PC . AB = 0$ ..............................(*a*).

Obtained from (i) (*a*) by substituting for $BC$, etc. in terms of distances from $P$.

$PA^2 . BC + PB^2 . CA + PC^2 . AB = -BC . CA . AB$ ...............(*b*).

Obtained from (i) (*a*) by substituting for $BC$, etc. in terms of distances from $P$.

$PA . PB . BA + PC . PA . AC + PB . PC . CB = BC . CA . AB$ .........(*c*).

Obtained from (ii) (*b*) by means of (ii) (*a*).

*Five points.* (iii)

$PA . QA . BC + PB . QB . CA + PC . QC . AB = -BC . CA . AB$

or $PA . DA . BC + PB . DB . CA + PC . DC . AB = -BC . CA . AB$ .........(*a*).

Subtract from (ii) (*b*) and, after removing factor $PQ$, the relation is reduced to (ii) (*a*).

$PA . BC . CD . DB + PB . CD . DA . AC + PC . DA . AB . BD + PD . AB . BC . CA = 0$ .........(*b*).

Substitute for products $AB.BC.CA$, etc. from (ii) (*b*).

$$PA^2.BC.CD.DB+PB^2.CD.DA.AC+PC^2.DA.AB.BD+PD^2.AB.BC.CA=0 \quad \text{.........}(c).$$

Substitute for the products $AB.BC.CA$, etc. from (ii) (*b*).

*Six points.* (iv)

$$PA.QA.BC.CD.DB+PB.QB.CD.DA.AC+PC.QC.DA.AB.BD$$
$$+PD.QD.AB.BC.CA=0 \quad \text{.........}(a).$$

Subtract from (iii) (*c*). Remove factor $PQ$ and the relation reduces to (iii) (*b*).

The preceding results may be easily proved analytically by means of determinants. Let $a$, $b$, $c$, $d$, ... be the distances of $P$ from the points $A$, $B$, $C$, $D$, .... Then

(iii) (*a*) $PA.DA.BC+PB.DB.CA+PC.DC.AB$

$$=a(d-a)(b-c)+b(d-b)(c-a)+c(d-c)(a-b)$$
$$=d\Sigma a(b-c)-\Sigma a^2(b-c)=d\begin{vmatrix} a & b & c \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$$
$$=+(a-b)(b-c)(c-a)=-AB.BC.CA.$$

(iii) (*b*) $$-\Sigma PA.BC.CD.DB=\Sigma a(b-c)(c-d)(d-b)=\begin{vmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \end{vmatrix}=0.$$

(iii) (*c*) $$-\Sigma PA^2.BC.CD.DB=\Sigma a^2(b-c)(c-d)(d-b)=\begin{vmatrix} a^2 & b^2 & c^2 & d^2 \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \end{vmatrix}=0.$$

(iv) (*a*) $$\Sigma PA.QA.BC.CD.DB=\begin{vmatrix} a(q-a) & b(q-b) & c(q-c) & d(q-d) \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \end{vmatrix}=0.$$

Other results may be obtained by this method. Thus

(1) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c).$$

Therefore $\Sigma PA^3.BC+AB.BC.CA.(PA+PB+PC)=0.$

(2) $$\begin{vmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}=0.$$

Therefore $\Sigma PA(PB+PC+PD)BC.CD.DB=0.$

### Determination of Position by means of Ratios.

If two points $A$ and $B$ are given, the position of any point $P$ on the line $AB$ is uniquely determined if the ratio $\frac{AP}{BP}$ is given. If the ratio $\frac{AP}{BP}$ is negative the point $P$ lies between $A$ and $B$. If it is positive, it lies on the other portions of the line $AB$. For different positions of $P$ the ratio $\frac{AP}{BP}$ can have any (real) value, positive or negative. The expression $\frac{AP}{BP}$ is termed *the ratio* of $P$ with respect to $A$ and $B$.

The point $P$, whose ratio is $p$, can be constructed as follows. Draw any line $AQ$ through $A$. On $AQ$ take a point $R$ such that $\frac{AQ}{RQ}=p$, the point $R$ being taken on the same side of $Q$ as $A$ or on the opposite side according as $p$ is positive or negative. Join $RB$. The line through $Q$ parallel to $RB$ meets $AB$ in the required point $P$.

If $p=1$ the point $R$ coincides with $A$, and the line through $Q$, parallel to $RB$, is parallel to $AB$. The point $P$ is then said to be at an infinite distance. In this case $AP=BP$. This may be interpreted to mean that the point $P$ is at such a distance from $A$ and from $B$ that the finite distance $AB$ may be neglected in comparison with these distances. The point $P$ in this case is termed *the point at infinity* on $AB$.

### 8. Measurement of Angles.

The angle between the line $a$ and the line $b$, measured from $a$ to $b$, may be denoted by $\widehat{ab}$ and the angle measured from $b$ to $a$ by $\widehat{ba}$. As a convention it is assumed that $\widehat{ab}=-\widehat{ba}$ so that

$$\widehat{ab}+\widehat{ba}=0.$$

If $a$, $b$, $c$ are any three concurrent straight lines,

$$\widehat{ab}+\widehat{bc}+\widehat{ca}=0.$$

Hence whatever be the relative position of three concurrent lines $a$, $b$, $o$,

$$\widehat{ab}=\widehat{ob}-\widehat{oa}.$$

This enables all the angles at a point to be expressed in terms of the angles made by lines through the point with a given line. The segment of the line $a$ from which $\overset{\frown}{ab}$ is measured must however be the same in every case. If $a'$ is the extension of $a$ through $ab$ then

$$\overset{\frown}{ab} + \overset{\frown}{ba'} = \pi.$$

If a circle be described with its centre at the point of intersection of given lines the angles at this point are measured (in circular measure) by the arcs cut off by these lines divided by the radius. Any homogeneous relation connecting the angles at this point may therefore be interpreted as a relation connecting arcs of a circle whose centre is at the point of intersection of the lines.

Thus if $\overset{\frown}{ab}$ be a length measured along the arc of a circle it is seen by the method of the last article that

$$\overset{\frown}{bc} . \overset{\frown}{ad} + \overset{\frown}{ca} . \overset{\frown}{bd} + \overset{\frown}{ab} . \overset{\frown}{cd} = 0,$$

and likewise that

$$\overset{\frown}{oa}^2 . \overset{\frown}{bc} + \overset{\frown}{ob}^2 . \overset{\frown}{ca} + \overset{\frown}{oc}^2 . \overset{\frown}{ab} = - \overset{\frown}{bc} . \overset{\frown}{ca} . \overset{\frown}{ab}.$$

The whole of the examples of the last article may be similarly interpreted.

### 9. Projection in a Plane.

If any number of points $A, B, C, D, E \ldots W \ldots$ situated on a straight line $u$ be joined to any point $S$—not on $u$—by straight lines $a, b, c, d, e \ldots w \ldots$ and these lines meet a second line $u'$ in points $A', B', C', D', E' \ldots$, then the points $A', B', C', D', E' \ldots$ are said to be the projections of the points $A, B, C, D, E \ldots$ from the centre of projection $S$ on the line $u'$.

Similarly the points $A', B', C', D', E' \ldots$ may be projected from another centre of projection $S'$ on another line $u''$ into points $A'', B'', C'', D'', E'' \ldots$.

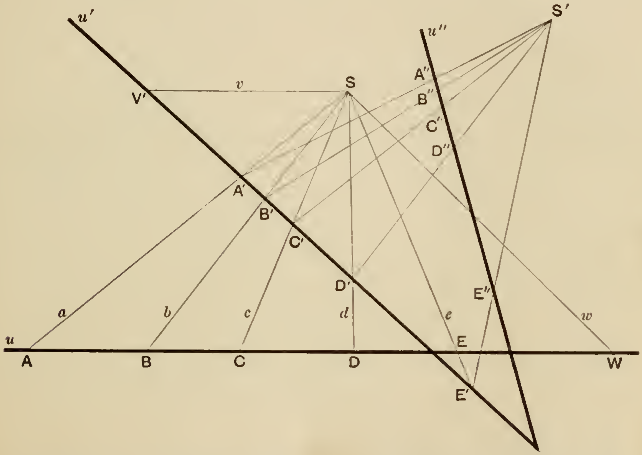
The ranges $ABCDE \ldots$, $A'B'C'D'E' \ldots$, $A''B''C''D''E'' \ldots$ are said to be projective as one is derived from the other by a series of projections and the points $AA'A''$, $BB'B''$, $CC'C''$, … are said to be corresponding points.

The ranges on $u$ and $u'$ which can be obtained by direct projection from each other are said to be in *Perspective.* Similarly the ranges on $u'$ and $u''$ are said to be in Perspective. The ranges on $u$ and $u''$ are projective but they are not in Perspective.

If through the point $S$ a straight line $w$ be drawn parallel to $u'$ it will meet $u$ in a point $W$. The projection of this point $W$ on the line $u'$ is the intersection of $w$ and $u'$. According to Euclid's definition these lines being parallel do not meet. In projective geometry they are regarded as meeting at an infinite distance and their point of intersection is termed *the point at infinity* on either of the lines. Thus when the ranges on $u$ and $u'$ are considered the point $W$, which corresponds to the point at infinity on $u'$, is obtained by drawing through $S$ a line parallel to $u'$ to meet $u$ at $W$.

Similarly the point $V'$ of the range $A'B'C'D'$ ... which corresponds to the point on $u$, which is at an infinite distance, is obtained by drawing through $S$ a line parallel to $u$ to meet $u'$ in $V'$. The points $W$ and $V'$ are called the *vanishing points* of the projective ranges on $u$ and $u'$.

The point at infinity on any line is considered to be at such a distance from all points, which are at a finite distance, that its distance from all such points may be regarded as being the same. Hereafter (Art. 22) it will be shown that all lines which are parallel may be regarded as meeting at one and the same point, which point is at an infinite distance.



A relationship which holds for the distances between the points $A, B, C, D, E$ ... is said to be a *projective relationship* if the same relationship necessarily holds for the corresponding distances between the corresponding points on any range derived from it by projection.

Likewise if an expression involving the distances between the points $A$, $B$, $C$, $D$, $E$ ... is such that the same expression, involving the corresponding distances on any range derived from the given one by a series of projections, has the same value, then that expression is said to be a *projective expression*.

To one point of a given range corresponds one and only one point—the corresponding point—on any other range derived from it by a series of projections.

The subject of projection is fully treated in Chapter IV.

### 10. Projective Forms. Simple Ratio.

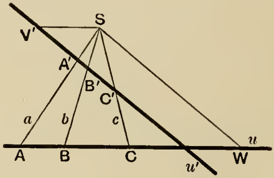
A simple ratio of the distances between three collinear points is not generally projective.

There are, however, two special cases that require consideration.

(1) If the cutting lines $u$ and $u'$ are parallel then

$$\frac{AC}{BC} = \frac{A'C'}{B'C'}$$

and the simple ratio is projective.



(2) Let the lines joining $ABC$ to $S$ meet $u'$ in $A'B'C'$.

Then from the triangle $ASC$,

$$\frac{AC}{AS} = \frac{\sin \widehat{ac}}{\sin SCA}.$$

Also from the triangle $BSC$, $\frac{BC}{BS} = \frac{\sin \widehat{bc}}{\sin SCA}$.

Therefore $\frac{AC}{BC} : \frac{AS}{BS} = \frac{\sin \widehat{ac}}{\sin \widehat{bc}}$ and similarly $\frac{A'C'}{B'C'} : \frac{A'S}{B'S} = \frac{\sin \widehat{ac}}{\sin \widehat{bc}}$.

The expression $\frac{AC}{BC} : \frac{AS}{BS}$ may be conveniently written $(ABC\,.\,S)$ and, from the above, it is seen that it is projective, when $S$ is the centre of projection, for $(ABC\,.\,S) = (A'B'C'\,.\,S)$.

*Elements at Infinity.* If $A$ be the point at infinity on $u$, which may be denoted by $\infty$, then $A'$ becomes the vanishing point $V'$ on $u'$. Similarly, if $B'$ be the point at infinity on $u'$, denoted by $\infty'$, $B$ becomes the vanishing point $W$ on $u$.

The relation $(ABC\,.\,S) = (A'B'C'\,.\,S)$ then becomes

$$(\infty\, WC\,.\,S) = (V'\infty' C'\,.\,S).$$

Hence

$$\frac{\infty C}{WC} : \frac{\infty S}{WS} = \frac{V'C'}{\infty' C'} : \frac{V'S}{\infty' S}.$$

Therefore, since $\infty C = \infty S$ and $\infty' C = \infty' S$,

$$\frac{WS}{WC} = \frac{V'C'}{V'S},$$

or

$$WC.V'C' = WS\,.\,V'S = \text{a constant}.$$

Hence *the product of the distances of two corresponding points of two projective ranges from the vanishing points of these ranges is constant.*

If $A_1$, $A_2$, $A_3 \ldots B$, $C$ are collinear points it can be shown that a relation of the form

$$l\frac{BA_1}{CA_1} + m\frac{BA_2}{CA_2} + n\frac{BA_3}{CA_3} + \ldots = 0$$

is projective.

Divide each term by $\frac{BS}{CS}$. Then the above relation becomes

$$l(BCA_1 . S) + m(BCA_2 . S) + n(BCA_3 . S) + \ldots = 0,$$

which is projective since by (2) each term is unaltered by projection from $S$.

## 11. Ratio of a Ratio. Anharmonic Ratio.

Given four collinear points the *ratio of the ratios* of their distances is subject to certain conditions projective. This is by far the most important deduction from (2), Art. 10.

Consider in Figure, Art. 9, the four collinear points $A, B, C, D$. Let the centre of projection be $S$.

Then $(ABC . S)$ and $(ABD . S)$ are projective.

Therefore
$$\frac{(ABC . S)}{(ABD . S)} = \frac{\frac{AC}{BC} : \frac{AS}{BS}}{\frac{AD}{BD} : \frac{AS}{BS}} = \frac{AC}{BC} : \frac{AD}{BD},$$

and the expression $\frac{AC}{BC} : \frac{AD}{BD}$ is projective.

When $A, B, C, D$ are collinear $\frac{AC}{BC} : \frac{AD}{BD}$ is written $(ABCD)$ and is called *the anharmonic ratio* (or one of the anharmonic ratios) of the four points $A, B, C, D$.

From Art. 10 (2) it is seen that

$$\frac{AC}{BC} : \frac{AS}{BS} = \frac{\sin \widehat{ac}}{\sin \widehat{bc}} \quad \text{and} \quad \frac{AD}{BD} : \frac{AS}{BS} = \frac{\sin \widehat{ad}}{\sin \widehat{bd}},$$

$$\therefore \quad (ABCD) = \frac{AC}{BC} : \frac{AD}{BD} = \frac{\sin \widehat{ac}}{\sin \widehat{bc}} : \frac{\sin \widehat{ad}}{\sin \widehat{bd}}.$$

The latter expression is sometimes written $(abcd)$ and is called the anharmonic ratio of the pencil $abcd$. The anharmonic ratio of the pencil formed by joining any four points $A$, $B$, $C$, $D$ to any point $P$ is denoted by $(P.ABCD)$ and every line cuts such a pencil in four points which have the same anharmonic ratio as the anharmonic ratio of the pencil. In some books $\dfrac{AC}{BC} : \dfrac{AD}{BD}$ is written as $(ACBD)$, *i.e.* the middle letters are interchanged.

*Examples*: If $A$, $B$, $C$, $D$, $C'$, $D'$ are collinear points,

(1) $\dfrac{(ABCD)}{(ABC'D)} = (ABCC')$,

(2) $\dfrac{(ABCD)}{(ABCD')} = (ABD'D)$,

(3) $(ABCD)(ABC'D') = (ABCD')(ABC'D)$,

(4) $(ABCD)(ABDD') = (ABCD')$.

### 12. Determination of the changes in the value of the Anharmonic Ratio arising from a change in the order in which the elements are taken.

Let $A$, $B$, $C$, $D$ be collinear points, then

$$(ABCD) = \frac{AC}{BC} : \frac{AD}{BD} = \lambda \text{ (suppose).}$$

(1)
$$(ABDC) = \frac{AD}{BD} : \frac{AC}{BC} = \frac{1}{\lambda},$$

$$(BACD) = \frac{BC}{AC} : \frac{BD}{AD} = \frac{1}{\lambda}.$$

Therefore *if the first and second or the third and fourth elements of an anharmonic ratio are interchanged the value of the anharmonic ratio is changed from* $\lambda$ *to* $\dfrac{1}{\lambda}$.

(2)
$$BC.AD + CA.BD + AB.CD = 0. \quad \text{(Art. 7.)}$$

$$\therefore\ 1 - \frac{AC.BD}{BC.AD} - \frac{AB.CD}{CB.AD} = 0,$$

$$\therefore\ 1 - (ABCD) - (ACBD) = 0,$$

$$\therefore\ (ACBD) = 1 - \lambda.$$

Similarly
$$(DBCA) = 1 - \lambda.$$

Therefore *if the second and third or the first and fourth elements of an anharmonic ratio are interchanged the value of the anharmonic ratio is changed from* $\lambda$ *to* $1 - \lambda$.

(3) $$\begin{aligned}(CBAD) &= 1-(CABD) && \text{by (2)},\\ &= 1-\frac{1}{(ACBD)} && \text{by (1)},\\ &= 1-\frac{1}{1-(ABCD)} && \text{by (2)},\\ &= 1-\frac{1}{1-\lambda}=\frac{\lambda}{\lambda-1}.\end{aligned}$$

Similarly $$(ADCB)=\frac{\lambda}{\lambda-1}.$$

Therefore *if the first and third or the second and fourth elements of an anharmonic ratio are interchanged the value of the anharmonic ratio is changed from* $\lambda$ *to* $\frac{\lambda}{\lambda-1}$.

If in (1), (2) or (3) after one pair of elements have been interchanged, the remaining pair are interchanged, the value of the anharmonic ratio is again $\lambda$. Hence the following important result is obtained.

*If the four elements, which make up an anharmonic ratio, are divided in any way into two pairs, and the elements in both these pairs are interchanged, the value of the anharmonic ratio is unaltered.*

It follows as an immediate consequence that if $\lambda$ is the anharmonic ratio $(ABCD)$ the values of the anharmonic ratios of the points $A, B, C, D$ taken in any different order must be either

$$\lambda,\quad \frac{1}{\lambda},\quad 1-\lambda,\quad \frac{1}{1-\lambda},\quad \frac{\lambda}{\lambda-1}\quad \text{or}\quad \frac{\lambda-1}{\lambda}.$$

Hence only six different values of the anharmonic ratios of four given points can be obtained by varying the order in which these points are taken.

*Given the anharmonic ratio of four points of a range and the position of three of the points—no two of which coincide—the fourth point is uniquely determined.*

Suppose that the three given points are $A$, $B$, $C$ and that the position of a fourth point $D$ has to be determined.

Then $$(ABCD)=\frac{AC}{BC}:\frac{AD}{BD}=\lambda\ \text{(suppose)},$$

therefore $$\frac{AD}{BD}=\frac{1}{\lambda}\frac{AC}{BC}=\text{a known quantity}.$$

Therefore $D$ is determined as the point which divides the distance between $A$ and $B$ in a given ratio, externally or internally, according as $\frac{1}{\lambda}\frac{AC}{BC}$ is positive or negative. Such a point by Art. 7 is uniquely determined.

From the preceding it may be deduced at once that *given the anharmonic ratio of a pencil of four rays and the position of three of the rays—no two of which coincide—the position of the fourth ray is uniquely determined.*

Hence *if the anharmonic ratios of two ranges or pencils are equal, and three pairs of corresponding elements coincide, the fourth pair must likewise coincide.*

### Coincidence of Points forming an Anharmonic Range.

*Given the position of three points $A$, $B$, $C$, no two of which coincide, to find the values of the anharmonic ratio $(ABCD)$ for which the point $D$ coincides with one of the three given points.*

If $(ABCD)=1$ then $\frac{AC}{BC}=\frac{AD}{BD}$. Therefore $C$ and $D$ coincide.

If $(ABCD)=0$ then $(ACBD)=1$. Therefore $B$ and $D$ coincide.

If $(ABCD)=\infty$ then $(BCAD)=1$. Therefore $A$ and $D$ coincide.

Hence *if the anharmonic ratio $(ABCD)$ equals $\infty$, 0 or 1, the point $D$ coincides with $A$, $B$, or $C$.* For other values the point $D$ is distinct from $A$, $B$, and $C$.

*If two out of four points $A$, $B$, $C$, $D$ coincide, to find the value of the anharmonic ratio $(ABCD)$ for all positions of the other two points.*

(1) Let $A$ and $B$ coincide, then

$$(AACD)=\frac{AC}{AC}:\frac{AD}{AD}=1 \text{ for all positions of } C \text{ and } D,$$

unless $C$ or $D$ coincide with $A\,(\equiv B)$ in which case the anharmonic ratio is indeterminate.

(2) Let $A$ and $C$ coincide, then

$$(ABAD)=\frac{AA}{BA}:\frac{AD}{BD}=0 \text{ for all positions of } B \text{ and } D,$$

unless $B$ or $D$ coincide with $A\,(\equiv C)$ in which case the anharmonic ratio is indeterminate.

(3) Let $A$ and $D$ coincide, then

$$(ABCA)=\frac{AC}{BC}:\frac{AA}{BA}=\infty \text{ for all positions of } B \text{ and } C,$$

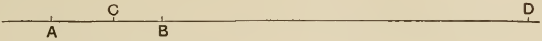
unless $B$ or $C$ coincide with $A\,(\equiv D)$ in which case the anharmonic ratio is indeterminate.

Hence *when two and only two points of a range coincide the value of the anharmonic ratio is either* 1, 0 *or* $\infty$ *and the other two points may occupy any positions.*

*Anharmonic ratio of four points three of which coincide.*

The anharmonic ratio in this case is indeterminate, but a meaning may be assigned to it in certain cases by considering the limiting process by which the three points come to coincide.

Let the points, which become coincident, be $A$, $B$, and $C$, and let the distances between $A$ and $C$ and between $C$ and $B$ be $ld$ and $md$, where $d$ in the limit is infinitely small.



Then
$$(ABCD)=\frac{AC}{BC}:\frac{AD}{BD}$$
$$=\frac{ld}{-md}:\frac{ld+CD}{CD-md}=-\frac{l}{m} \text{ when } d \text{ is infinitely small.}$$

Therefore, wherever the point $D$ is situated, the anharmonic ratio is $-\frac{l}{m}$. An important instance of this will arise hereafter.

**Sign of an Anharmonic Ratio.**

When the points $A$, $B$, $C$, $D$ occur in this order on a straight line, the anharmonic ratio $(ABCD)=\frac{AC}{BC}:\frac{AD}{BD}$ is positive.

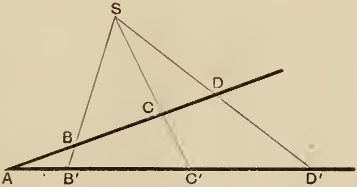
If any one of the points moves along the line till it coincides with one of the other points the value of the anharmonic ratio becomes either zero, infinity or unity. In the two former cases the anharmonic ratio, when the points cross, changes its sign but not in the last case.

Thus if the four points be divided into two groups, viz.: (1) $A$ and $B$ and (2) $C$ and $D$, a change in the relative position of the points in either group does not affect the sign of the anharmonic ratio, but any alteration of the position of the points in one group relative to those in the other causes the sign of the anharmonic ratio to change.

Hence *if the points $A$, $B$, $C$, $D$ are situated on a straight line in the order in which the letters occur, the anharmonic ratio $(ABCD)$ is positive and every change of position of $A$ or $B$ with respect to $C$ or $D$ changes the sign of the anharmonic ratio.*

## Anharmonic condition that three lines should be concurrent.

*If through a point $A$ two lines $ABCD$ and $AB'C'D'$ be drawn and the anharmonic ratio $(ABCD)$ equals the anharmonic ratio $(AB'C'D')$ then the three lines $BB'$, $CC'$, $DD'$ are concurrent.*



Let $BB'$ and $CC'$ meet at $S$. Join $SD$ meeting $AB'C'$ in $D''$.

Then
$$(AB'C'D')=(ABCD)$$
$$=(AB'C'D''),$$

$$\therefore \frac{AC'}{B'C'} : \frac{AD'}{B'D'} = \frac{AC'}{B'C'} : \frac{AD''}{B'D''},$$

$$\therefore \frac{AD'}{B'D'} = \frac{AD''}{B'D''}, \quad \therefore D'' \text{ coincides with } D' \text{ (Art. 7).}$$

### Anharmonic condition that three points should be collinear.

*If through two points on a line $a$, three sets of lines $bcd$ and $b'c'd'$ be drawn and the anharmonic ratio of the pencil $(abcd)$ be equal to the anharmonic ratio of the pencil $(ab'c'd')$ then the points $bb'$, $cc'$, $dd'$ are collinear.*

Let $s$ the line joining $bb'$ and $cc'$ meet $d$ in $D$ and let the line joining $D$ to $b'c'$ be $d''$.

Then $$(ab'c'd') = (abcd) = (ab'c'd'').$$

Hence the lines $d'$ and $d''$ coincide.

### Projection of a Simple Ratio.

The statement at the beginning of Art. 10 to the effect that "a simple ratio of the distances between three collinear points is not generally projective" is subject to an important qualification. Consider three collinear points $A$, $B$, $C$. On the line $ABC$ there is a point at infinity, which may be regarded as equally distant from every point of the figure which is at a finite distance. If this point be denoted by $\infty$

$$\frac{AC}{BC} = \frac{AC}{BC} : \frac{A\infty}{B\infty} = (ABC\infty).$$

If $A$, $B$, $C$, $\infty$ are projected from any centre into $A'B'C'V'$ then

$$\frac{AC}{BC} = (ABC\infty) = (A'B'C'V')$$

$$= \frac{A'C'}{B'C'} : \frac{A'V'}{B'V'}.$$

By means of this fact theorems may often be generalised.

*Similar ranges.* Two ranges are said to be *similar* when their points at infinity are corresponding points. In this case

$$(ABC\infty) = (A'B'C'\infty),$$

$$\therefore \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

Hence corresponding segments are proportional. If two pairs of corresponding segments are proportional then every pair will be proportional, and the ranges are similar.

*Equal ranges.* Two ranges are said to be equal when the corresponding segments are equal. When the corresponding segments have the same sign the ranges are said to be *directly equal*; when they have opposite signs, the ranges are said to be *oppositely equal.* If two pairs of corresponding segments are equal, all corresponding segments are equal and the ranges are equal.

*Equal pencils.* Two pencils are said to be equal when the angles between pairs of corresponding rays are equal. When the angles are equal and have the same sign the pencils are said to be *directly equal.* When the signs are different the pencils are said to be *oppositely equal.* If two pairs of angles between corresponding rays are equal, the pencils are equal.

The student may be helped in recognising projective ranges by the fact that two ranges are projective

(1) When the sum, the difference, the ratio or the product of the distances of corresponding points from two fixed points on the bases is constant,

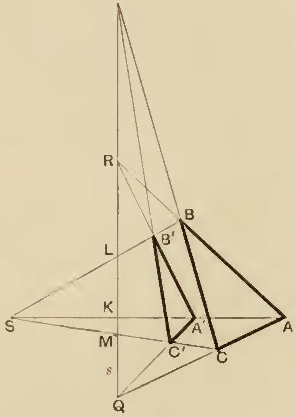
or (2) When pairs of corresponding points subtend equal angles at some fixed point or points.

## 13. Application of Anharmonic Ratios to Important Theorems of Projective Geometry and Plane Perspectives.

(*a*) *If the lines joining the three vertices of two triangles in pairs are concurrent at a point* ($S$) *then the corresponding sides intersect in pairs on a straight line* ($s$).

Let $AA'$, $BB'$, $CC'$, the connectors of the vertices of the triangles, meet in $S$. Let $AC$, $A'C'$ intersect in $Q$ and $AB$ and $A'B'$ in $R$. Let $QR$ be $s$, and let $AA'$, $BB'$, $CC'$ meet $s$ in $K$, $L$ and $M$. Then

$$(SMC'C) = (SKA'A) = (SLB'B).$$

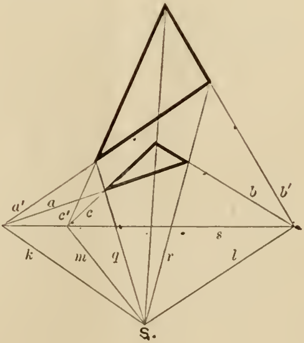


Hence the lines $ML$, $B'C'$, $BC$ are concurrent, and therefore, $B'C'$ and $BC$ intersect in a point on $s$.

*Conversely and correlatively.*

*If the sides of two triangles intersect in pairs on a straight line* ($s$) *then the lines joining the corresponding vertices are concurrent at a point* ($S$).

Let $aa'$, $bb'$, $cc'$, the pairs of corresponding sides of the triangles, intersect on $s$. Let the line joining



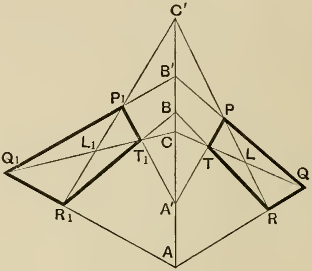
$ac$ and $a'c'$ be $q$ and that joining $ab$ and $a'b'$ be $r$. Let $q$ and $r$ intersect in $S$. Join $aa'$, $bb'$, $cc'$ to $S$ by $k$, $l$ and $m$.

Then $$(smc'c) = (ska'a) = (slb'b).$$

Hence the points $ml$, $b'c'$, $bc$ are collinear, and therefore the line joining $b'c'$ to $bc$ passes through $lm$ or $S$.

The triangles in the above case are said to be in plane Perspective. $S$ is termed the centre and $s$ the axis of Perspective.

(*b*) *If five pairs of sides of two quadrangles intersect on a straight line, then the sixth pair intersect on the same straight line.*



Let the quadrangles be $PQRT$ and $P_1Q_1R_1T_1$ and let $PR$, $PQ$, $RT$, $QT$ and $PT$ intersect $P_1R_1$, $P_1Q_1$, $R_1T_1$, $Q_1T_1$, and $P_1T_1$ respectively in $C'B'BCA'$ on the line $s$.

It is necessary to prove that $A$ the point of intersection of $QR$ and $Q_1R_1$ lies on $s$.

Since the sides of the triangles $QTP$ and $Q_1T_1P_1$ intersect in pairs on $s$, these triangles are in perspective, and $PP_1$, $QQ_1$, $TT_1$ are concurrent at some point $S$. Similarly the triangles $PRT$ and $P_1R_1T_1$ are in perspective and the lines $PP_1$, $RR_1$, $TT_1$ are concurrent at the same point $S$. Therefore, since $RR_1$, $QQ_1$, $TT_1$ are concurrent at $S$, the triangles $QTR$ and $Q_1T_1R_1$ are in perspective. Therefore the points $B$, $C$ and $A$ are collinear, and $A$ lies on ($s$).

This theorem may be proved directly as follows. Let $QR$ and $Q_1R_1$ meet $s$ in $A_1$ and $A_2$, then in the figure

$$(T.A'CBC') = (T_1.A'CBC'),$$
$$\therefore\ (PLRC') = (P_1L_1R_1C'),$$
$$\therefore\ (Q.PLRC') = (Q_1.P_1L_1R_1C'),$$
$$\therefore\ (B'CA_1C') = (B'CA_2C').$$

Hence, since $B'CC'$ are collinear, $A_1$ and $A_2$ must coincide in a point $A$ on $s$.

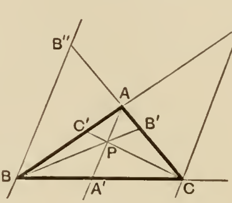
The correlative theorem, which may be proved by similar methods, is as follows.

*If five pairs of vertices of two quadrilaterals are collinear with a given point, then the sixth pair are also collinear with this point.*

From this it follows that if two quadrangles $ABCD$ and $A'B'C'D'$ are such that five of the sides of one are parallel to the five corresponding sides of the other, then the sixth pair of sides are also parallel. The axis of Perspective is the line at infinity.

(*c*) **Ceva's Theorem.** *If the lines joining any point $P$ to the vertices $ABC$ of a triangle meet the opposite sides in $A'B'C'$ respectively, then*

$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = -1.$$



Draw through $B$ and $C$ lines parallel to $AA'$ to meet $CA$ and $BA$ respectively in $B''$ and $C''$. In this case $\infty$, the point at infinity on $AA'$, is also the point at infinity on $BB''$ and $CC''$. Then

$$(B . AA'P\infty) = (C . AA'P\infty),$$

$$\therefore\ (ACB'B'') = (ABC'C''),$$

by taking intercepts on $AC$ and $AB$.

$$\therefore\ \frac{AB'}{CB'} : \frac{AB''}{CB''} = \frac{AC'}{BC'} : \frac{AC''}{BC''},$$

$$\therefore\ \frac{AB'}{CB'} \cdot \frac{BC'}{AC'} = \frac{AB''}{CB''} : \frac{AC''}{BC''} = \frac{A'B}{CB} : \frac{A'C}{BC} = -\frac{BA'}{CA'},$$

$$\therefore\ \frac{AB'}{CB'} \cdot \frac{BC'}{AC'} \cdot \frac{CA'}{BA'} = -1.$$

*Conversely. If points $A'B'C'$ are taken on the sides of a triangle $ABC$ such that*

$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = -1,$$

*then the lines $AA'$, $BB'$, $CC'$ are concurrent.*

Let the lines $BB'$, $CC''$ intersect in $P$, and let $AP$ meet $BC$ in $A''$. Then

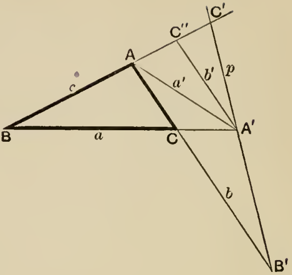
$$\frac{BA''}{CA''} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = -1,$$

$$\therefore\ \frac{BA'}{CA'} = \frac{BA''}{CA''} \text{ (from data)},$$

$$\therefore\ A'' \text{ coincides with } A'.$$

(*d*) **Menelaus' Theorem.** *If any straight line meets the sides of a triangle $ABC$ in points $A'B'C'$, then*

$$\frac{BA'}{CA'} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1.$$



Let the sides of the triangle be $a$, $b$, $c$ and let the straight line be $p$. Through $A'$ draw $b'$, parallel to $b$, to meet $c$ in $C''$.

Let $AA'$ be $a'$ and let the point at infinity on $b'$ be $\infty$.

Then $(b\,.\,aa'pb') = (c\,.\,aa'pb')$,

$$\therefore\ (CAB'\infty) = (BAC'C'')$$

by taking intercepts on $b$ and $c$.

$$\therefore\ \frac{CB'}{AB'} = \frac{BC'}{AC'} : \frac{BC''}{AC''},$$

$$\therefore\ \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = \frac{AC''}{BC''} = \frac{CA'}{BA'},$$

$$\therefore\ \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} \cdot \frac{BA'}{CA'} = 1.$$

*Conversely. If points $A'B'C'$ are taken on the sides of a triangle $ABC$ such that*

$$\frac{CB'}{AB'} \cdot \frac{AC'}{BC'} \cdot \frac{BA'}{CA'} = 1,$$

*then the points $A'B'C'$ are collinear.*

Let the line joining $B'C'$ meet $a$ in $A''$. Then

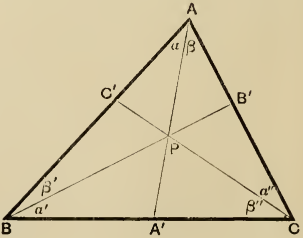
$$\frac{BA''}{CA''} \cdot \frac{CB'}{AB'} \cdot \frac{AC'}{BC'} = 1,$$

$$\therefore\ \frac{BA'}{CA'} = \frac{BA''}{CA''} \text{ (from data)},$$

$\therefore$ $A''$ coincides with $A'$.

**Secondary form of Ceva's Theorem.**

Let the sides of the triangle be denoted by $a$, $b$, $c$, and the lines $AP$, $BP$, $CP$ by $a_1$, $b_1$, $c_1$.



Then $$\frac{BA'}{CA'} = \frac{c \sin \widehat{ca_1}}{b \sin \widehat{ba_1}}$$

with similar expressions for

$$\frac{CB'}{AB'} \text{ and } \frac{AC'}{BC'}.$$

Therefore by substituting in Ceva's relation

$$\frac{\sin \widehat{ca_1}}{\sin \widehat{ba_1}} \cdot \frac{\sin \widehat{ab_1}}{\sin \widehat{cb_1}} \cdot \frac{\sin \widehat{bc_1}}{\sin \widehat{ac_1}} = -1$$

or, denoting the angles as in the figure and disregarding the signs,

$$\frac{\sin \alpha \sin \alpha' \sin \alpha''}{\sin \beta \sin \beta' \sin \beta''} = +1.$$

### 14. Analytical Expressions for Anharmonic Ratios.

If two points $B$ and $C$ are given, the position of any other point $A$ on the line joining them is determined by the ratio $\frac{BA}{CA}$. It is sometimes a convenience to speak of this as the ratio of the point $A$ and to denote it by some expression such as $x$.

If $B$ and $C$ are given points and $x_1$, $x_2$, $x_3$, $x_4$ are the ratios of four other points $A_1$, $A_2$, $A_3$, $A_4$ on the line $BC$, then

(1) $$(BCA_1A_2)=\frac{x_1}{x_2},$$

(2) $$(A_1A_2A_3A_4)=\frac{(x_3-x_1)(x_4-x_2)}{(x_3-x_2)(x_4-x_1)}.$$

This result follows at once from the fact that

$$A_1A_2=a\frac{x_1-x_2}{(x_1-1)(x_2-1)} \text{ where } a=BC.$$

(3) If $x_1$, $x_2$ are the roots of the equation $a'x^2+2h'x+b'=0$ and $x_3$, $x_4$ the roots of the equation $a''x^2+2h''x+b''=0$, then

$$[2h'h''-a'b''-a''b']^2=4\left[\frac{1+\lambda}{1-\lambda}\right]^2(h'^2-a'b')(h''^2-a''b'')$$

where $\lambda$ is a value of the anharmonic ratio $(A_1A_2A_3A_4)$.

(4) If $x_1$, $x_2$, $x_3$, $x_4$ are the roots of the equation $ax^4+4bx^3+6cx^2+4dx+e=0$ and $\lambda$ is any one of the anharmonic ratios of $A_1A_2A_3A_4$, then

$$\frac{1}{3^3}\frac{(\lambda+1)^2(\lambda-\frac{1}{2})^2(\lambda-2)^2}{(\lambda^2-\lambda+1)^3}=\frac{\{c^3-ace+ad^2-2bcd+b^2e\}^2}{\{3c^2+ae-4bd\}^3}.$$

## EXAMPLES.

(1) If $(ABCD)=\lambda$, $\frac{1-\lambda}{AB}=\frac{1}{AD}-\frac{\lambda}{AC}$.

(2) If $(ABPD)=(ABCQ)$ show that $(PQAC)=(PQDB)$.

(3) If $E$ and $F$ are two fixed points and the anharmonic ratio

$$(AA'EF)=(BB'EF)=(CC'EF)=(DD'EF)$$

show that $$(ABCD)=(A'B'C'D').$$

(4) If four collinear points $A$, $B$, $P$, $Q$ are such that $(ABPQ)$ is constant and $A$ and $B$ are fixed, then the ranges described by $P$ and $Q$ are projective.

(5) If four concurrent lines $a$, $b$, $p$, $q$ are such that $(abpq)$ is constant and $a$ and $b$ are fixed, then the pencils described by $p$ and $q$ are projective.

# CHAPTER III

## PROJECTIVE FORMS HARMONIC

### 15. Harmonic Ranges and Pencils.

In Chapter II, Art. 12, the case of four points whose anharmonic ratio has the value $\infty$, 0 or 1 was considered and it was proved that two out of the four points must coincide. In this chapter another important case of particular values of the anharmonic ratio of four points will be considered.

The range formed by the four points $A$, $B$, $C$, $D$ is said to be *Harmonic* when

$$(ABCD) = (ABDC).$$

By Art. 12, $$(ABDC) = \frac{1}{(ABCD)}.$$

Therefore in this case $$(ABCD) = \frac{1}{(ABCD)}.$$

Hence $$(ABCD)^2 - 1 = 0,$$

and therefore $$(ABCD) = 1 \text{ or } -1.$$

If $(ABCD) = 1$ it is seen, by Art. 12, that two of the four points coincide. Therefore if the points are distinct

$$(ABCD) = -1.$$

Also (Art. 12) in this case

$$(ACBD) = 2 \text{ and } (CABD) = \tfrac{1}{2}.$$

Hence, if the anharmonic ratio of four points is either $-1$, 2 or $\frac{1}{2}$, the four points form a Harmonic range.

The points $C$ and $D$, which may be interchanged without altering the values of the above anharmonic ratios, are said to be Harmonic Conjugates of $A$ and $B$. But since

$$(ABCD) = (CDAB) \text{ and } (ABDC) = (CDBA)$$

therefore $$(CDAB) = (CDBA).$$

Hence $A$ and $B$ may likewise be interchanged without altering the value of the anharmonic ratio, and are therefore also called harmonic conjugates of $C$ and $D$.

Thus a Harmonic Range consists of two pairs of points. These pairs of points are interchangeable, as also are the points in each pair. The points in each pair of harmonic conjugates divide the distance between the other pair, internally and externally, in the same ratio, for if

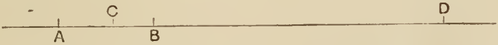
$$(ABCD) = -1, \text{ then } \frac{AC}{BC} = -\frac{AD}{BD}.$$

If $C$ and $D$ are harmonic conjugates of $A$ and $B$ and $D$ is at infinity, $C$ bisects the distance between $A$ and $B$, and conversely, if $C$ bisects the distance between $A$ and $B$, the range made up of $A$, $B$, $C$ and the point at infinity on the line is a harmonic range. Consequently the projection of this range on any line is also a harmonic range.

If four points $A$, $B$, $C$, $D$ forming a harmonic range be joined to any point $S$ by lines $a$, $b$, $c$, $d$, it follows from Art. 11 that $(abcd) = -1$ and the pencil $a$, $b$, $c$, $d$ is then said to be a *harmonic pencil.* It is obvious that any harmonic pencil is cut by any transversal in a harmonic range.

### Case of Coincident Elements of a Harmonic Range.

In Art. 12 it was proved that if two and only two of the elements of any anharmonic range coincide the value of the anharmonic range must be either $\infty$, 0 or 1. Therefore, if two elements of a harmonic range coincide, a third point must likewise coincide in the same point. Hence a pair of conjugate points must coincide with one of the other pair of conjugates.



It has likewise been shown that if

$$AC = l \, . \, d \text{ and } CB = m \, . \, d$$

$$(ABCD) = -\frac{l}{m}$$

in the limit when $d$ is infinitely small, and that $D$ in this case may be any point at a finite distance from $C$. If $l = m$ the range is harmonic.

Hence *if two points $A$ and $B$ equally distant from a point $C$ are supposed to move uniformly towards the point $C$ and in the limit coincide with it, the range formed by $ABC$ and any other point on the line forms in the limit a harmonic range of which $A$ and $B$ are a pair of conjugate points.*

### 16. Properties of Harmonic Ranges and Pencils.

If $O$ be the middle point between two conjugates $A$ and $B$ of a harmonic range $ABCD$ and $o$ be a line bisecting the angle $\widehat{ab}$ between a pair of conjugate lines of a harmonic pencil $abcd$, the chief properties of Harmonic Ranges and Pencils may be stated as follows:

(a) $\dfrac{2}{AB}=\dfrac{1}{AC}+\dfrac{1}{AD}$, (a′) $2\cot\widehat{ab}=\cot\widehat{ac}+\cot\widehat{ad}$,

(b) $CA\,.\,CB=CO\,.\,CD$, (b′) $2\sin\widehat{ca}\,.\,\sin\widehat{cb}=\sin 2\widehat{co}\,.\,\tan\widehat{cd}$,

(c) $OC\,.\,OD=OA^2=OB^2$, (c′) $\cot\widehat{oc}\,.\,\cot\widehat{od}=\cot^2\widehat{oa}=\cot^2\widehat{ob}$,

(d) $\dfrac{OC}{OD}=\left(\dfrac{AC}{AD}\right)^2=\left(\dfrac{BC}{BD}\right)^2$, (d′) $\dfrac{\sin 2\widehat{oc}}{\sin 2\widehat{od}}=\left(\dfrac{\sin\widehat{ca}}{\sin\widehat{da}}\right)^2=\left(\dfrac{\sin\widehat{cb}}{\sin\widehat{db}}\right)^2$.

The results on the left-hand side may be deduced from the fact that

$$\frac{AC}{BC}:\frac{AD}{BD}=-1,$$

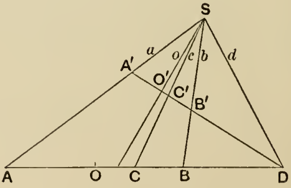
and those on the right-hand either from

$$\frac{\sin\widehat{ac}}{\sin\widehat{bc}}:\frac{\sin\widehat{ad}}{\sin\widehat{bd}}=-1,$$

or from the corresponding relation on the left-hand.

(a) Express all the distances in $(ABCD)=-1$ in terms of distances from $A$. Then after simplification it is seen that

$$\frac{2}{AB}=\frac{1}{AC}+\frac{1}{AD}.$$



(a′) Let $S$ be the vertex of the pencil. Drop a perpendicular $DB'C'A'$ on $a$, meeting $c$ and $b$ in $C'$ and $B'$ (not shown in figure). Then $A'B'C'D$ is harmonic.

Hence $$2\frac{1}{A'B'}=\frac{1}{A'C'}+\frac{1}{A'D'},$$

$$\therefore\ 2\frac{SA'}{A'B'}=\frac{SA'}{A'C'}+\frac{SA'}{A'D'},$$

$$\therefore\ 2\cot\widehat{ab}=\cot\widehat{ac}+\cot\widehat{ad}.$$

(b) From (a) since $2\,.\,CO=CA+CB$,

$$\frac{2}{CD}=\frac{1}{CA}+\frac{1}{CB}=\frac{CA+CB}{CA\,.\,CB}=\frac{2\,.\,CO}{CA\,.\,CB},$$

$$\therefore\ CO\,.\,CD=CA\,.\,CB.$$

(b′) Similarly, since $2\widehat{co}=\widehat{ca}+\widehat{cb}$,

$$2\cot\widehat{cd}=\cot\widehat{ca}+\cot\widehat{cb}=\frac{\sin 2\,.\,\widehat{co}}{\sin\widehat{ca}\,.\,\sin\widehat{cb}},$$

$$\therefore\ 2\sin\widehat{ca}\sin\widehat{cb}=\sin 2\widehat{co}\tan\widehat{cd}.$$

(*c*) Since
$$\frac{AC}{CB}=\frac{AD}{BD},$$
$$\therefore\ \frac{AC-CB}{AC+CB}=\frac{AD-BD}{AD+BD},$$
$$\therefore\ \frac{2.OC}{2.OB}=\frac{2.OB}{2.OD},$$
$$\therefore\ OC.OD=OB^2=OA^2.$$

(*c′*) Drop a perpendicular from *D* on *o* meeting *a*, *b*, *c* and *o* in *A′B′C′O′*. Then *O′* is the middle point of *AB*, therefore, since *A′B′C′D* is harmonic,
$$O'C'.O'D=O'A'^2,$$
$$\therefore\ \frac{SO'}{O'C'}\cdot\frac{SO'}{O'D}=\left(\frac{SO'}{O'A'}\right)^2,$$
$$\therefore\ \cot\overset{\frown}{oc}\cot\overset{\frown}{od}=\cot^2\overset{\frown}{oa}=\cot^2\overset{\frown}{ob}.$$

(*d*) From (*c*)
$$\frac{OA}{OC}=\frac{OD}{OA}=\frac{AD}{-AC}$$
by subtracting numerators and denominators.
$$\therefore\ \frac{OD}{OC}=\left(\frac{AD}{AC}\right)^2.$$

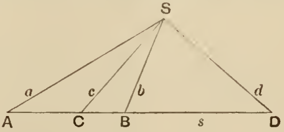
Similarly
$$\frac{OD}{OC}=\left(\frac{BD}{BC}\right).$$

(*d′*) From (*c′*)
$$\frac{\cot\overset{\frown}{oc}}{\cot\overset{\frown}{oa}}=\frac{\cot\overset{\frown}{oa}}{\cot\overset{\frown}{od}}=\frac{\sin\overset{\frown}{ca}}{-\sin\overset{\frown}{da}}:\frac{\sin\overset{\frown}{oc}}{\sin\overset{\frown}{od}},$$
$$\therefore\ \frac{\cot\overset{\frown}{oc}}{\cot\overset{\frown}{od}}=\left(\frac{\sin\overset{\frown}{ca}}{\sin\overset{\frown}{da}}\right)^2:\left(\frac{\sin\overset{\frown}{oc}}{\sin\overset{\frown}{od}}\right)^2,$$
$$\therefore\ \frac{\sin 2.\overset{\frown}{oc}}{\sin 2.\overset{\frown}{od}}=\left(\frac{\sin\overset{\frown}{ca}}{\sin\overset{\frown}{da}}\right)^2.$$

Similarly
$$\frac{\sin 2.\overset{\frown}{oc}}{\sin 2.\overset{\frown}{od}}=\left(\frac{\sin\overset{\frown}{cb}}{\sin\overset{\frown}{db}}\right)^2.$$

*The internal and external bisectors of any angle form with the lines that include the angle a harmonic pencil;* and conversely,

*If two conjugate rays of a harmonic pencil are at right angles they are the internal and external bisectors of the angle formed by the other pair of conjugate lines.*



Let any line *s* meet the lines *a* and *b* in *A* and *B*, and *c* and *d* the bisectors of the angle $\overset{\frown}{ab}$ in *C* and *D*.

Then from the triangle *ASB*,
$$\frac{AC}{BC}=\frac{AS}{BS}=-\frac{AD}{BD}.\quad\text{(Euclid VI 3 and A.)}$$

Hence the range *ABCD* is harmonic.

Conversely, if the pencil $abcd$ is harmonic and the rays $c$ and $d$ are at right angles draw $o$ to bisect the angle $ASB$.

Then from ($b'$) $\quad 2\sin\widehat{ca}\,.\,\sin\widehat{cb} = \sin 2\,\widehat{co}\,.\,\tan\widehat{cd}.$

But, since $\widehat{cd} = \frac{\pi}{2}$, $\tan\widehat{cd}$ is infinitely great while $\sin\widehat{ca}\,.\,\sin\widehat{cb}$ is finite.

Therefore $\quad \sin 2\,\widehat{co} = 0$ and consequently $\widehat{co} = 0$.

Hence the line $c$ coincides with $o$ and is a bisector of the angle $\widehat{ab}$. Since $d$ and $c$ are at right angles $d$ is the other bisector.

### 17. Analytical Expressions in connexion with Harmonic Ranges.

Adopting the notation of Art. 14 and denoting the ratios of the points $A_1$, $A_2$, $A_3$, $A_4$ with reference to the points $B$, $C$ by the letters $x_1$, $x_2$, $x_3$, $x_4$ respectively, the following results are obtained :

(1) If $\frac{x_1}{x_2} = -1$ the range $BCA_1A_2$ is harmonic, therefore the points $x_1$ and $-x_1$ are harmonic conjugates of $B$ and $C$.

(2) If $x_1$ and $x_2$ are the roots of the equation $ax^2+b=0$ then the range $BCA_1A_2$ is harmonic.

(3) The condition that the four points whose ratios are $x_1$, $x_2$, $x_3$, $x_4$ should form a harmonic range is

$$(x_1+x_2)(x_3+x_4)=2x_1x_2+2x_3x_4.$$

(4) The condition that $A_3$, $A_4$ should be harmonic conjugates of $B$ and $A_2$ is

$$x_2(x_3+x_4)=2x_3x_4.$$

(5) If $x_1$, $x_2$ are given as roots of the equation $a'x^2+2h'x+b'=0$ and $x_3$, $x_4$ as roots of the equation $a''x^2+2h''x+b''=0$ then the condition that $A_1$, $A_2$ should be harmonic conjugates of $A_3$, $A_4$ is

$$a''b'+a'b''-2h'h''=0.$$

(6) If $x_1$, $x_2$, $x_3$, $x_4$ are the roots of the equation $ax^4+4bx^3+6cx^2+4dx+e=0$ the condition that the range $A_1A_2A_3A_4$ should be harmonic is

$$c^3-ace+ad^2-2bcd+b^2e=0.$$

(7) The common harmonic conjugates of $B$, $C$ and $A_1$, $A_2$ are the points

$$+\sqrt{x_1x_2} \text{ and } -\sqrt{x_1x_2}.$$

## EXAMPLES.

(1) If $C$, $D$ be harmonic conjugates of $A$, $B$ and $P$ be any point on the line $ABCD$, then

$$2\,.\,PA\,.\,PB+2\,.\,PC\,.\,PD=(PA+PB)(PC+PD)$$

and, conversely, if this relation holds $ABCD$ is harmonic.

Express $(ABCD)=-1$ in terms of distances from any point $P$ on the line and the required result is obtained.

*(2) If $Q$ and $Q'$ be the harmonic conjugates of $P$ with respect to $AB$ and $CD$ respectively, all the points being collinear, show that $ABCD$ is harmonic if

$$\frac{1}{PA\,.\,PB}+\frac{1}{PC\,.\,PD}=\frac{2}{PQ\,.\,PQ'}.$$

Since $Q$ is a harmonic conjugate of $P$ with respect to $AB$, $\frac{2}{PQ}=\frac{1}{PA}+\frac{1}{PB}$; similarly $\frac{2}{PQ'}=\frac{1}{PC}+\frac{1}{PD}$. Therefore

$$\frac{2}{PA\,.\,PB}+\frac{2}{PC\,.\,PD}=\left(\frac{1}{PA}+\frac{1}{PB}\right)\left(\frac{1}{PC}+\frac{1}{PD}\right)$$

or $2\,.\,PC\,.\,PD+2\,.\,PA\,.\,PB=(PB+PA)(PD+PC)$. Hence from (1) the result follows.

(3) If $C_1$, $C_2$, $C_3$ be the middle points of $X_1Y_1$, $X_2Y_2$, $X_3Y_3$, which are Harmonic Conjugates of $AB$, and $P$ be any point on the line $AB$, then

$$C_2C_3\,.\,PX_1\,.\,PY_1+C_3C_1\,.\,PX_2\,.\,PY_2+C_1C_2\,.\,PX_3\,.\,PY_3=0.$$

If $C$ be the middle point of $AB$, then by (1) $PX_1\,.\,PY_1+PA\,.\,PB=2PC_1\,.\,PC$.

Substituting in the given relation it becomes $\Sigma C_2C_3(2\,.\,PC_1\,.\,PC-PA\,.\,PB)=0$, or $2PC\Sigma C_2C_3\,.\,PC_1-PA\,.\,PB\,.\,\Sigma C_2C_3=0$.

Hence for the given relation to hold $\Sigma C_2C_3\,.\,PC_1$ or $\Sigma(PC_3-PC_2)PC_1$ must be zero, but this is the case.

(4) If $E$, $F$ be the middle points of $AB$, $CB$ respectively, two segments of a straight line, which cut each other harmonically, then

$$EF^2=EB^2+FD^2.$$

Describe circles on $AB$ and $CD$ as diameters cutting at $P$. Then

$$EP^2=EB^2=EC\,.\,ED.$$

Hence the circles cut at right angles and therefore

$$EF^2=EP^2+PF^2=EB^2+FD^2.$$

*Note.* From this a solution is obtained of the problem "to place two given segments so that their ends may form a harmonic range."

(5) If $AB$ be harmonic conjugates of $XY$ and of $PQ$, then

$$\frac{PX}{QX}\cdot\frac{PY}{QY}=\left(\frac{PA}{QA}\right)^2.$$

(6) If $ABCD$ be a harmonic range and $P$ any point on the line

$$2\frac{PB}{AB}=\frac{PC}{AC}+\frac{PD}{AD}.$$

(7) If $AB$ be harmonic conjugates of $XY$ and of $PQ$, then

$$XP\,.\,QA\,.\,AY=-YQ\,.\,PA\,.\,AX.$$

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* See also Examples (3) and (6), Chapter VIII.

(8) If $AA'$ be harmonic conjugates of $BB'$ and $P$ any point on the line prove that

$$(AA'BP) = -(AA'B'P).$$

(9) $A$, $B$, $C$, $D$ are four collinear points, determine $P$ and $Q$ which are harmonic conjugates of $AB$ and $CD$ and prove that

$$AC.AD : BC.BD :: AP^2 : BP^2.$$

(10) If $ABCD$ be a harmonic range $A$ and $B$ being conjugate and $C$ and $D$, then

$$\text{(i)} \quad AB.CD = 2AD.CB,$$

$$\text{(ii)} \quad 2OR.OR' = OP.OP' + OQ.OQ',$$

where $O$ is any point on the line and $PP'$, $QQ'$, $RR'$ are the mid-points of $BC$, $AD$, $CA$, $BD$, $AB$ and $CD$.

(11) If $A'$, $B'$, $C'$, $D'$ be harmonic conjugates of $A$, $B$, $C$, $D$ with respect to $EF$, prove that $(A'B'C'D') = (ABCD)$.

(12) $A$, $B$, $A'$, $B'$ are four points on a straight line, $P$, $P'$ the harmonic conjugates of any point in the line with regard to $AB$, $A'B'$; prove that $P$, $P'$ describe homographic ranges and determine the self-corresponding points and the vanishing points of the ranges.

(13) $A$ and $A'$ are any pair of harmonic conjugates of $E$ and $F$ and $P$ and $Q$ any pair of fixed points on the line $EF$. Prove that

$$\frac{PA.QA}{EA.FA} + \frac{PA'.QA'}{EA'.FA'} \text{ is constant.}$$

Express all lengths as distances from $O$ the middle point of $EF$. Reduce to one fraction remembering that $OA.OA' = OE^2 = OF^2$, then the factor $2OE^2 - OA^2 - OA'^2$ can be removed from numerator and denominator, and the given expression becomes

$$\frac{OE^2 - OP.OQ}{OE^2}.$$

(14) If $(ABKD) = (ABCL)$, where $A$, $B$, $C$, $D$, $K$, $L$ are collinear points, then

$$\frac{KA}{LA} \cdot \frac{KB}{LB} = \frac{KC}{LC} \cdot \frac{KD}{LD}.$$

# CHAPTER IV

## CONICAL PROJECTION AND PLANE PERSPECTIVE

**18.** There are four kinds of Projections which are in general use, viz.,

(1) Orthogonal Projection,
(2) Projection in a Plane,
(3) Conical Projection,
(4) Homology, or Plane Perspective.

In addition to the above, it is also possible to project from any given axis. (1) and (2) are particular cases of (3), and (4) may also be derived from (3).

The fundamental theorems of Conical Projection and of Homology or Plane Perspective are similar, although in some cases they have to be proved in different ways. In Conical Projection the proofs are generally simpler and bring home more directly to the learner the reason why the theorems are true. In Homology, on the other hand, there is a simplification in the fact that the figures lie in one plane, and the learner is not confronted with the difficulty of comprehending figures in space. The fundamental theorems of Conical Projection and of Homology will be given side by side together with their proofs. Afterwards, it will be shown how the latter may be deduced from the former.

### Conical Projection.

*General method of obtaining one plane figure from another plane figure by Conical Projection.*

Let the given figure consisting of points $A$, $B$, $C$, ... and lines $a$, $b$, $c$, ... be situated in the plane $\sigma$.

Take any point $S$ outside the plane $\sigma$.

Construct the lines $SA$, $SB$, $SC$, ... and the planes $Sa$, $Sb$, $Sc$, ....

Consider any plane $\sigma'$ (not coincident with $\sigma$). The lines $SA$, $SB$, $SC$, ... will intersect this plane in points, say, $A'$, $B'$, $C'$, .... The planes $Sa$, $Sb$, $Sc$, ... will intersect this plane in lines, say, $a'$, $b'$, $c'$, ....

The figure made up of the points $A'$, $B'$, $C'$, ... and the lines $a'$, $b'$, $c'$, ... in the plane $\sigma'$ is said to be a projection upon the plane $\sigma'$ of the original figure made up of the points $A$, $B$, $C$, ... and the lines $a$, $b$, $c$, ....

The points $A'$, $B'$, $C'$, ... in one figure are said to correspond to the points $A$, $B$, $C$, ... in the other, and the lines $a'$, $b'$, $c'$, ... in one figure are said to correspond to the lines $a$, $b$, $c$, ... in the other.

It will be seen that every pair of corresponding points are collinear with $S$, and, since any two corresponding lines $a$ and $a'$ are the intersections of a plane through $S$ with the planes $\sigma$ and $\sigma'$, the lines $a$ and $a'$ meet the line $\sigma\sigma'$ in the same point.

### Homology, or Plane Perspective.

*General method of obtaining one plane figure from another plane figure by Homology or Plane Perspective.*

Let the given figure consisting of points $A$, $B$, $C$, ... and lines $a$, $b$, $c$, ... be situated in the plane $\sigma$. Take any point $S$ outside the plane $\sigma$.

On any plane $\sigma'$, not coincident with $\sigma$, construct with centre $S$ a projection of the given figure, consisting of points $A'$, $B'$, $C'$, ... and of lines $a'$, $b'$, $c'$, ....

Take any other centre of projection $S'$. From $S'$ project the figure consisting of points $A'$, $B'$, $C'$, ... and lines $a'$, $b'$, $c'$, ... upon the plane $\sigma$, so as to form a figure in the plane $\sigma$ consisting of the points $A_1$, $B_1$, $C_1$, ... and lines $a_1$, $b_1$, $c_1$, ....

The figure consisting of the points $A_1$, $B_1$, $C_1$, ... and the lines $a_1$, $b_1$, $c_1$, ... is said to be in plane perspective with the figure consisting of the points $A$, $B$, $C$, ... and the lines $a$, $b$, $c$, .... The points $A_1$, $B_1$, $C_1$, ... in one figure are said to correspond to the points $A$, $B$, $C$, ... in the other, and the lines $a_1$, $b_1$, $c_1$ in one figure are said to correspond to the lines $a$, $b$, $c$ in the other.

Every pair of corresponding points $A$, $A_1$ are collinear with the point where $SS'$ meets $\sigma$, for these points are situated on the line in which the plane $SS'A'$ meets $\sigma$.

Any two corresponding lines $a$ and $a_1$ meet $\sigma\sigma'$ in the point where it is met by $a'$. Therefore all pairs of corresponding lines intersect on the fixed line $\sigma\sigma'$.

Hence two figures in plane perspective are such that every pair of corresponding points are collinear with a given point, termed the *centre of Perspective*, and every pair of corresponding lines intersect on a given line, termed the *axis of Perspective.*

In practice, these theorems are used to construct figures in plane perspective and the consideration of figures in space is thus excluded.

**19.** From the preceding it follows that both in Conical Projection and in Plane Perspective

(1) To points correspond points and to straight lines correspond straight lines.

(2) To collinear points correspond collinear points and to concurrent lines correspond concurrent lines.

(3) Corresponding curves are the loci of corresponding points and corresponding envelopes are the envelopes of corresponding lines.

(4) To points or lines in one figure which coincide in the limit correspond in the other figure points or lines which likewise coincide in the limit.

(5) To a point of intersection of a line and a curve corresponds a point of intersection of the corresponding line and curve.

(6) A line meets a curve in the same number of points as the corresponding line meets the corresponding curve. Hence a curve and its corresponding curve are said to be of the same *degree.*

(7) To a tangent to a curve at a point corresponds the tangent at the corresponding point to the corresponding curve.

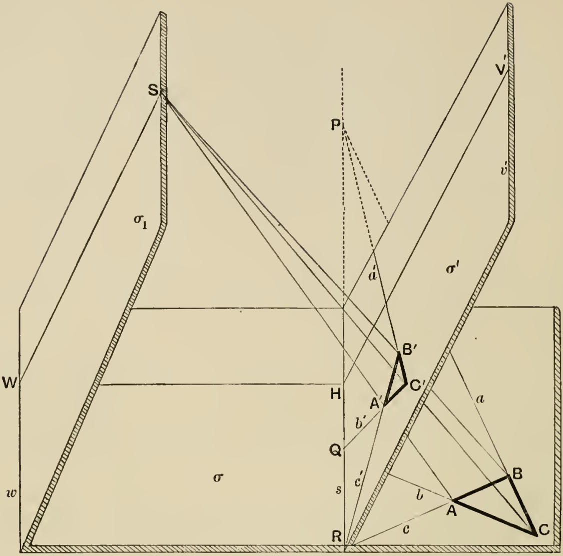
(8) To the tangents from any point to a curve correspond the tangents from the corresponding point to the corresponding curve. Hence a curve and its corresponding curve are said to be of the same *class.*

(9) The anharmonic ratio of any four collinear points in one figure is equal to the anharmonic ratio of the corresponding points of the other figure. For the two ranges are in plane perspective (see Art. 9) with $S$ for centre of perspective.

(10) A harmonic range is projected into a harmonic range.

(11) Either figure may be deduced from the other by the same process.

These points are more fully illustrated in the following article.



### 20. Construction of figures by Conical Projection.

Let the given figure of which $A$, $B$, $C$ are three points be in the plane $\sigma$ (Figure page 34).

Take $S$ any point outside the plane $\sigma$.

Join $S$ to $A$, $B$ and $C$, and on $SA$, $SB$, $SC$ take any points $A'$, $B'$, $C'$. These points lie in some plane $\sigma'$.

Since $AA'$ and $BB'$ meet in $S$, $AB$ and $A'B'$ lie in a plane and meet in some point $R$.

Similarly, $BC$ and $B'C'$ meet at a point $P$ and $AC$ and $A'C'$ meet at a point $Q$.

### Construction of figures in Plane Perspective.

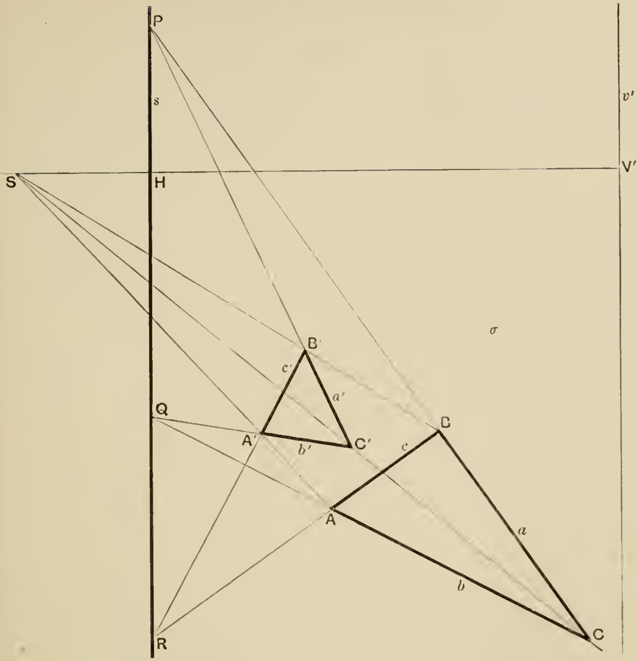
Let the given figure of which $A$, $B$, $C$ are three points be in the plane $\sigma$ (Figure page 35).

Take $S$ any point in the plane $\sigma$.

Join $S$ to $A$, $B$ and $C$, and on $SA$, $SB$, $SC$ take any points $A'$, $B'$, $C'$. These points lie in the plane $\sigma$.

Since $AA'$ and $BB'$ are in the plane $\sigma$, $AB$ and $A'B'$ meet in some point $R$.

Similarly, $BC$ and $B'C'$ meet at a point $P$ and $AC$ and $A'C'$ meet at a point $Q$.



But $P$, $Q$ and $R$ must lie on the line of intersection of $\sigma$ and $\sigma'$ and are therefore collinear.

The line $PQR$ or $\sigma\sigma'$ is usually denoted by the letter $s$.

The figure obtained by projection on the plane $\sigma'$ is now completely determined, for the point corresponding to any given point $D$ in the plane $\sigma$ can be found.

But the triangles $ABC$ and $A'B'C'$ are such that the lines joining their vertices are concurrent, and therefore $P$, $Q$ and $R$ (Art. 13 (*a*)) are collinear.

The line $PQR$ is usually denoted by the letter $s$ and is called the axis of homology or perspective.

The figure in perspective with the given one is now completely determined, for the point corresponding to any given point $D$ can be found, since (Art. 18) corresponding points are collinear with $S$ and corresponding lines intersect on $s$.

For join $D$ to $S$.

Then the point corresponding to $D$ is that in which $SD$ meets the plane $\sigma'$.

### 21. Determining elements of a Conical Projection.

Given the figure to be projected, the determining elements of the projection may be:

(1) The point $S$ and any three points $A'$, $B'$, $C'$ on $SA$, $SB$, $SC$ respectively which are to correspond to $A$, $B$, $C$.

(2) The point $S$ and the plane $\sigma'$, or, what is equivalent thereto,

$S$ and $s$ together with $A'$, the point corresponding to $A$, which must be collinear with $S$ and $A$.

or correlatively,

$s$ and $S$ together with $a'$, the line corresponding to $a$, which must be concurrent with $s$ and $a$.

(3) Any two points $A'$ and $B'$ which correspond to $A$ and $B$, such that $AA'$ and $BB'$ meet at some point $S$, and any point $C'$ on $SC$ which corresponds to $C$.

*Note:* Even if $C'$ is not given, a point $R$ on $s$ is determined and the plane $\sigma'$ for different positions of $C'$ will rotate round the line $RA'B'$.

or correlatively,

any two lines $a'$ and $b'$, which correspond to $a$ and $b$, such that $aa'$ and $bb'$ lie on a line $s$, and any line $c'$ which corresponds to $c$ and intersects $c$ on $s$.

*Note:* Even if $c'$ is not given, the line of intersection of the planes $aa'$ and $bb'$ passes through $S$, and the point $S$ will move along this line of intersection, for different positions of $c'$.

For join $D$ to $S$. Let $DA$ meet $s$ in $T$.

Then the point $D'$ corresponding to $D$ is that in which $SD$ meets $TA'$. For $DA$ and $D'A'$, being corresponding lines, must intersect on $s$.

### Determining elements of figures in Homology or Perspective.

Given the figure whose perspective is required, the determining elements of the perspective may be:

(1) The point $S$ and any three points $A'$, $B'$, $C'$ on $SA$, $SB$, $SC$ respectively which are to correspond to $A$, $B$, $C$.

(2)

$S$ and $s$ together with $A'$, the point corresponding to $A$, which must be collinear with $S$ and $A$.

or correlatively,

$s$ and $S$ together with $a'$, the line corresponding to $a$, which must be concurrent with $s$ and $a$.

(3) Any two points $A'$ and $B'$ which correspond to $A$ and $B$, such that $AA'$ and $BB'$ determine some point $S$, and any point $C'$ on $SC$ which corresponds to $C$.

*Note:* Even if $C'$ is not given, a point $R$ on $s$ is determined and the line $s$ for different positions of $C'$ will rotate round the point $R$.

or correlatively,

any two lines $a'$ and $b'$, which correspond to $a$ and $b$, such that $aa'$ and $bb'$ determine a line $s$, and any line $c'$ which corresponds to $c$ and intersects $c$ on $s$.

*Note:* Even if $c'$ is not given, the line joining $ab$ to $a'b'$ must pass through $S$, and the point $S$ will move along this line, for different positions of $c'$.

(4) Any two points $A'$ and $B'$ corresponding to $A$ and $B$, and such that $AA'$ and $BB'$ intersect, and the direction of $s$ in the plane $\sigma$.

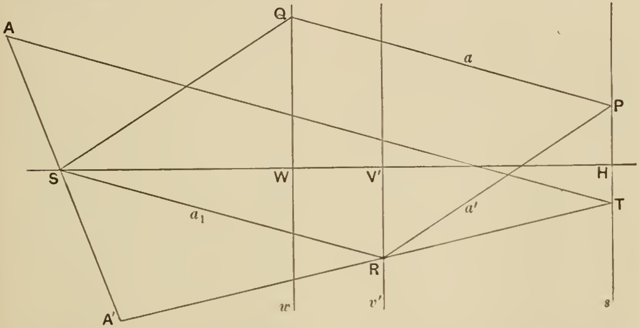
(4) Any two points $A'$ and $B'$ corresponding to $A$ and $B$, and the direction of $s$.

*Note:* For another method of construction see Art. 28.

In each of the above cases the point $S$, and the plane $\sigma'$ or the axis of perspective $s$, can be found.

## 22. Vanishing Lines.

*The loci of the points in one figure which correspond to the points at infinity in the other are straight lines termed the vanishing lines. There is one vanishing line in each figure.*



To find the point (see Fig. page 34) corresponding in the second figure to the point at infinity on a line $a$, draw through $S$ a line $a_1$ parallel to $a$. This line will meet $\sigma'$ in the point $a_1\sigma'$. Similarly the points corresponding to the points at infinity on $b, c, d, \ldots$ are obtained as the points of intersection of the plane $\sigma'$ with the lines $b_1, c_1, d_1, \ldots$ drawn through $S$ parallel to $b, c, d \ldots$. These lines lie in a plane through $S$ parallel to $\sigma$ and the line of intersection of this plane with $\sigma'$ gives the locus of the points corresponding to the points at infinity on $\sigma$. This line is parallel to $s$, and is termed the *vanishing line*.

To find the point (see Fig. page 37) corresponding in the second figure to the point at infinity on a line $a$, draw through $S$ a line $a_1$ parallel to $a$. If $A'$ corresponds to $A$ and a parallel to $a$ be drawn through $A$ meeting $s$ in $T$, then the required point is given as the intersection of $a_1$ and $TA'$. By considering the points corresponding to the points at infinity on any other lines $b, c, d, \ldots$ any number of points on the locus may be obtained. This locus by Addendum I is a straight line parallel to $s$, and is termed the *vanishing line*.

This vanishing line may be conveniently denoted by $v'$.

Similarly the points at infinity in the figure $\sigma'$ correspond to the points lying on a vanishing line $w$, which is obtained by drawing a plane through $s$ parallel to $\sigma'$.

As the points at infinity in either figure correspond to points on a vanishing line, it is convenient to regard all the points at infinity in either figure as being on a straight line termed *the line at infinity*. Thus in the first figure the points at infinity are regarded as being on a line $v$ which corresponds to $v'$ and in the second on a line $w'$ which corresponds to $w$.

The *vanishing point* of a line is the point where the line meets the vanishing line in its figure. This point corresponds to the points at infinity on the corresponding line.

*A system of parallel lines in one figure corresponds to a system of lines in the other which intersect in a point on a vanishing line.*

For the above construction gives the same point on the vanishing line as corresponding to the point at infinity on each line of the system.

### Determination of Position of Vanishing Lines.

Through $S$ draw a plane perpendicular to $s$ meeting

$s$ in $H$,
$w$ in $W$,
$v'$ in $V'$.

This vanishing line may be conveniently denoted by $v'$.

Similarly the points at infinity in the second figure correspond to the points lying on a vanishing line $w$, which is also parallel to the axis of Perspective $s$.

As the points at infinity in either figure correspond to points on a vanishing line, it is convenient to regard all the points at infinity in either figure as being on a straight line termed *the line at infinity*. Thus in the first figure the points at infinity are regarded as being on a line $v$ which corresponds to $v'$ and in the second on a line $w'$ which corresponds to $w$.

The *vanishing point* of a line is the point where the line meets the vanishing line in its figure. This point corresponds to the points at infinity on the corresponding line.

*A system of parallel lines in one figure corresponds to a system of lines in the other which intersect in a point on a vanishing line.*

For the above construction gives the same point on the vanishing line as corresponding to the point at infinity on each line of the system.

### Determination of Position of Vanishing Lines.

Through $S$ draw a line perpendicular to $s$ meeting

$s$ in $H$,
$w$ in $W$,
$v'$ in $V'$.

Then $SWHV'$ is a parallelogram (Figure page 34).

Therefore

$$SW = V'H \text{ and } SV' = WH.$$

Therefore *the distance of either vanishing line from the centre of Projection* ($S$) *is equal to that of the other vanishing line from the line of intersection of the planes* ($s$).

Hence, given $S$ and any two of the lines $s$, $w$ and $v'$, the remaining line is completely determined.

**23.** The connector of $S$ with the vanishing point of any line $a$, viz. $(S.wa)$ in the figure $\sigma$, is parallel to the corresponding line $a'$. For $(S.wa)$ passes through $w'a'$ the point at infinity on $a'$.

Hence a line in one figure is parallel to the line joining $S$ to the vanishing point of the corresponding line in the other figure.

As an immediate consequence the following important result is obtained.

*The angle subtended by the vanishing points of two lines at $S$ is equal to the angle between the corresponding lines.*

Let $a$ and $a'$ be two corresponding lines which therefore meet in a point $P$ on $s$.

Let $a$ meet $w$ in $Q$ and $a'$ meet $v'$ in $R$ (Figure page 37).

Then $aw$ (or $Q$) corresponds to the point at infinity on $a'$. Therefore $SQ$ is parallel to $a'$. Similarly $SR$ is parallel to $a$. Therefore $SQPR$ is a parallelogram, and

$$SW = V'H \text{ and } SV' = WH.$$

Therefore *the distance of either vanishing line from the centre of Perspective* ($S$) *is equal to that of the other from the axis of Perspective* ($s$).

Hence, given $S$ and any two of the lines $s$, $w$ and $v'$, the remaining line is completely determined.

The connector of $S$ with the vanishing point of any line $a$, viz. $(S.wa)$ in one figure, is parallel to the corresponding line $a'$. For $(S.wa)$ passes through $w'a'$ the point at infinity on $a'$.

Hence a line in one figure is parallel to the line joining $S$ to the vanishing point of the corresponding line in the other figure.

As an immediate consequence the following important result is obtained.

*The angle subtended by the vanishing points of two lines at $S$ is equal to the angle between the corresponding lines.*

### 24. Nature of the figure obtained by Conical Projection.

If the centre of Projection $S$, the plane $\sigma$, and the vanishing line $w$ are given, the plane $\sigma'$ may move parallel to itself, and the lines $s$ and $v'$ will then move parallel to $w$ and at a constant distance apart, equal to the distance of $S$ from $w$.

In this case the Projections will be similar. For sections of a cone by parallel planes are similar.

The linear dimensions of figures obtained by projection are proportional to the perpendicular from the vertex on the plane of section or to the intercepts by these planes on any line through $S$ drawn in a given direction. The intercept $SV'$ is such a length. But

$$SV' = WH.$$

### Nature of the figure obtained by Plane Perspective.

If the centre of Perspective $S$, the first figure, and the vanishing line $w$ are given, the lines $s$ and $v'$ may move parallel to $w$ and will be at a constant distance apart, equal to the distance of $S$ from $w$.

In this case the Perspective figures are similar. For if $a$ and $b$, any two lines of the given figure, meet $w$ in $aw$ and $bw$, then the corresponding lines in every figure in perspective are parallel to the lines joining these points to $S$. Hence in all the perspective figures the angles between pairs of lines which correspond to a given pair of lines in the original figure are equal. Therefore the figures are similar.

Let $v'$ and $s$ be the vanishing line and axis of perspective of one figure, and $v_1'$ and $s_1$ be the vanishing line and axis of perspective of another figure.

Let two lines of the given figure be $a$ and $b$. Through $S$ draw lines parallel to $a$ and $b$ meeting $v'$ and $v_1'$ in $P'$ and $P_1'$ and in $Q'$ and $Q_1'$. These points correspond to the points at infinity on $a$ and $b$. Hence, if the central line meets $v'$, $s$, $v_1'$, $s_1$ in $V'$, $H$, $V_1'$ and $H_1$,

$$\frac{P'Q'}{P_1'Q_1'} = \frac{SV'}{S'V_1'} = \frac{WH}{WH_1}.$$

Therefore the linear dimensions of the similar figures obtained by projection are proportional to $WH$, the distance of the vanishing line from the intersection of the planes $\sigma$ and $\sigma'$.

Hence the nature of a projection depends on the position of the plane $(S.w)$ and the position of $S$ in this plane.

By taking any line in $\sigma$ as $w$ (the vanishing line) that line may be projected into the line at infinity.

**25. To form a figure by Projection such that the angles between the two pairs of lines, which correspond to two given pairs of lines, shall have given values, and the Projection of a given line shall be the line at infinity.**

Let $a$ and $b$, $c$ and $d$ be the lines the projections of which are to contain given angles $\alpha$ and $\beta$. Take as vanishing line of the figure $w$, the line which is to be projected to infinity. Draw any plane $\sigma_1$ through $w$. Construct the vanishing points $aw$, $bw$, $cw$, $dw$ of the lines $a$, $b$, $c$, $d$. These will be on the line of intersection $w$ of the planes $\sigma_1$ and $\sigma$.

In the plane $\sigma_1$ describe on $aw$, $bw$ a segment of a circle to contain an angle $\alpha$ and on $cw$, $dw$ describe a segment of a circle to contain an angle $\beta$.

Therefore the linear dimensions of the similar figures obtained by plane perspective are proportional to $WH$, the distance of the vanishing line from the axis of perspective $s$.

Hence the nature of the perspective figure depends on the position of $w$ in the plane and on the position of $S$ relative to it.

By taking any line in the plane as $w$ (the vanishing line) that line may be made to correspond to the line at infinity in the perspective figure.

**To form a figure by Plane Perspective such that the angles between the two pairs of lines, which correspond to two given pairs of lines, shall have given values, and the line corresponding to a given line shall be the line at infinity.**

Let $a$ and $b$, $c$ and $d$ be the lines whose corresponding lines are to contain given angles $\alpha$ and $\beta$. Take as vanishing line of the figure $w$, the line which is to correspond to the line at infinity. Construct the vanishing points $aw$, $bw$, $cw$, $dw$ of the lines $a$, $b$, $c$, $d$.

On $aw$, $bw$ describe a segment of a circle to contain an angle $\alpha$ and on $cw$, $dw$ describe a segment of a circle to contain an angle $\beta$.

Let these segments intersect in $S$[*].

If $S$ be taken as centre of projection and any plane $\sigma'$ parallel to $\sigma_1$ as the plane of section, then the lines $a'$, $b'$, $c'$, $d'$ corresponding to $a, b, c, d$ will be parallel to the lines joining $S$ to $aw$, $bw$, $cw$ and $dw$.

Therefore the angles between $a'$ and $b'$, $c'$ and $d'$ will be of the given magnitudes $\alpha$ and $\beta$. Also the given line $w$ will be projected to infinity. Hence the figure obtained by this Projection meets the given conditions.

Let these segments intersect in $S$[*].

If $S$ be taken as centre of perspective and any line $s$ parallel to $w$ as axis of perspective, $a'$, $b'$, $c'$, $d'$ the lines corresponding to $a, b, c, d$ will be parallel to the lines joining $S$ to $aw$, $bw$, $cw$ and $dw$.

Therefore the angles between $a'$ and $b'$, $c'$ and $d'$ will be of the given magnitudes $\alpha$ and $\beta$. Also the given line $w$ will be projected to infinity. Hence the figure obtained by this Plane Perspective meets the given conditions.

### 26. Connexion between Conical Projection and Plane Perspective.

In the preceding, the fundamental theorems of conical projection and of plane perspective have been proved independently. The latter, however, may be deduced from the former.

Suppose that the figures in the planes $\sigma$ and $\sigma'$ are given and likewise the correspondence between them. Let the plane $\sigma'$ rotate about the line of intersection of the planes $\sigma$ and $\sigma'$ (i.e. $s$). Let $a$, $b$, $c$ and $a'$, $b'$, $c'$ be any three pairs of corresponding lines which intersect in pairs of corresponding points $A$, $B$, $C$ and $A'$, $B'$, $C'$ (see Fig. page 34). When $\sigma'$ is rotated round $\sigma\sigma'$, the lines $aa'$, $bb'$, $cc'$ will continue to intersect on $\sigma\sigma'$. Hence the lines $AA'$, $BB'$, $CC'$ will continue to intersect in some point $S$. The pairs of points $AA'$, $BB'$ may be regarded as fixed and determining the position of $S$ and $CC'$ as any pair of corresponding points. Hence the figures, when their planes are rotated, will continue to have a centre of projection $S$.

Consider the parallelogram $SWHV'$. Because the figure in $\sigma'$ is given and likewise the correspondence, $HV'$ is constant. Therefore $WS$

---

[*] If the segments $aw$, $bw$ and $cw$, $dw$ overlap, the circles described thereon will intersect in a pair of real points and $S$ is real. If they do not overlap, the circles, in certain cases, will not intersect and in this case $S$ will be an imaginary point. The projection is then said to be imaginary. The criterion as to whether or not the projection or perspective is real is given in Art. 30.

is constant and since $W$ is fixed, $S$ describes a circle round $W$ in the plane perpendicular to the lines $w$, $v'$, $s$.

If the plane $\sigma'$ be rotated round $s$ till $\sigma$ and $\sigma'$ coincide, the two figures will be superposed in plane perspective in the plane $\sigma$ with $S$ as centre of perspective at a distance $HV'$ from $w$, and $s$ for axis of perspective. The rotation may take place in either of two ways, so that $S$ may lie on either side of $w$, and $v'$ on either side of $H$.

**27. Any two plane quadrangles may be derived the one from the other by a series of projections.**

Let $ABCD$ and $A'B'C'D'$ be the quadrangles situated in the same or different planes.

Through $D$ draw a plane distinct from the plane $ABCD$ and project the quadrangle $A'B'C'D'$ into a quadrangle in this plane so that the point corresponding to $D'$ coincides with $D$. The problem then reduces to the following:

"Given two planes $\sigma$ and $\sigma'$, three points $A, B, C$ in the former, three points $A', B', C'$ in the latter and $D$ a point on their line of intersection, to derive $A'B'C'D$ from $ABCD$ by projection."

There are two ways of proceeding:

(*a*) Through $D$ draw a line $s$ in the plane $\sigma$ to meet $BC$, $CA$ and $AB$ in $K$, $L$, and $M$, and a line $s'$ in the plane $\sigma'$ to meet $B'C'$, $C'A'$ and $A'B'$ in $K'$, $L'$ and $M'$.

Since $s$ and $s'$ intersect at $D$ they lie in a plane $\sigma_1$, in which $KK'$, $LL'$, $MM'$ form a triangle $A_1B_1C_1$.

The triangle $ABC$ is projective with the triangle $A_1B_1C_1$ since the sides intersect in $K$, $L$, $M$ on the line $\sigma\sigma_1$, and, since $D$ is on $\sigma\sigma_1$, the point $D$ corresponds to itself.

Similarly the triangle $A'B'C'$ is projective with the triangle $A_1B_1C_1$ since the sides intersect in $K'$, $L'$, $M'$ on the line $\sigma'\sigma_1$, and, since $D$ is on $\sigma'\sigma_1$, the point $D$ corresponds to itself.

Therefore the points $A'$, $B'$, $C'$, $D$ can be derived from the points $A$, $B$, $C$, $D$ by two projections.

(*b*) Let $AB$ meet $DC$ in $E$ and $A'B'$ meet $DC'$ in $E'$.

Since $DCE$ and $DC'E'$ are in one plane, $CC'$ and $EE'$ will meet in a point $S_1$. Draw a plane $\sigma_1$ through $DCE$ and project $A'B'C'D$ from the point $S_1$ on the plane $\sigma_1$. The points $D$, $C'$, $E'$ will thus be projected into the points $D$, $C$, $E$, and $A'$ and $B'$ will be projected into points $A''$, $B''$ in the plane $\sigma_1$, which are collinear with $E$.

Since the lines $ABE$ and $A''B''E$ are concurrent, $AA''$ and $BB''$ are concurrent at some point $S_2$ and from this point the quadrangle $A''B''CD$ can be projected into the quadrangle $ABCD$.

This result should be compared with that obtained in Art. 65, where it is shown that any two quadrangles can be placed in a plane so as to be in plane perspective.

*If three triangles situated in different planes are two by two in perspective and have the same axis of perspective, their three centres of perspective are collinear.*

Let the three triangles be (1) $ABC$, (2) $A'B'C'$, (3) $A''B''C''$. Then since the triangles are two by two in perspective and have the same axis of perspective, any three pairs of corresponding sides, say, $AB$, $A'B'$, $A''B''$, meet at a point on $s$ the common axis of perspective.

Let $S$ be the centre of perspective of (1) and (2),
$S'$ „ „ „ (2) and (3),
$S''$ „ „ „ (3) and (1).

Then the three points $S$, $S'$ and $S''$ are situated in the plane $AA'A''$, and also in the plane $BB'B''$. Hence they are all on the line of intersection of the planes $AA'A''$ and $BB'B''$ and are collinear. It is seen also that $CC'C''$ passes through this line.

The Correlative theorem is:

*If three triangles situated in different planes are two by two in perspective and have the same centre of perspective, their three axes of perspective are concurrent.*

### 28. Anharmonic ratios in connexion with figures in Plane Perspective.

*If $A$ and $A'$ be any two corresponding points of two figures in plane perspective, and $AA'$ meets $s$ in $P$, then if $S$ be the centre of perspective the anharmonic ratio $(SPAA')$ is constant.*

Take any other pair of corresponding points $B$ and $B'$. Let $BB'$ meet $s$ in $Q$. Then the lines $AB$ and $A'B'$ intersect in some point $R$ on $s$. Hence $(SPAA') = (SQBB') =$ a constant, for every pair of corresponding points. From this it follows that

(1) If $S$, $s$ and the anharmonic ratio $(SPAA')$ be given, the Perspective is completely determined. $(SPAA')$ may be termed the anharmonic ratio of the Perspective.

(2) If $A$ be taken at infinity, $(SPAA') = 1 : \frac{SA'}{PA'} = \frac{PA'}{SA'}$. In this case, $A'$ is any point on the vanishing line, which may be denoted by $V'$, and its locus is clearly a line parallel to $s$ such that

$$\frac{HV'}{SV'} = (SPBB') = \text{the anharmonic ratio of the perspective.}$$

(3) Similarly $\frac{HW}{SW} = (SPB'B) = \frac{1}{(SPB'B)} = \frac{SV'}{HV'}$;

$$\therefore\ \frac{HS+SW}{SW} = \frac{SH+HV'}{HV'};$$

$$\therefore\ SW = V'H.$$

## 29. Particular cases of Conical Projection and Plane Perspective.

*Conical Projection.*

(1) Let $s$ be at infinity. In this case, the cutting planes are parallel and the figures are similar. The vanishing lines coincide with the line at infinity and the ratio of any pair of corresponding segments is constant.

(2) Let $S$ be at infinity. In this case, the cone formed by joining points in the figures to $S$ is replaced by a cylinder or prism. The line at infinity corresponds to itself. Orthogonal projection is a particular instance of this case.

*Plane Perspective.*

(1) Let $s$ be at infinity. Then the figures are said to be similar and similarly placed. The vanishing lines coincide with the line at infinity and $S$ is termed the centre of similitude.

(2) Let $S$ be at infinity. Then the line joining any pair of corresponding points is parallel to a fixed direction. The line at infinity corresponds to itself. Such figures are termed "homological by affinity."

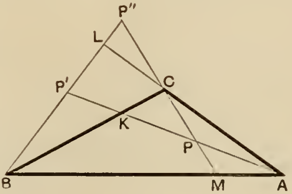
(3) Let the anharmonic ratio of the plane perspective be $-1$. The Perspective is then said to be harmonic. In this case $(SXAA')=(SXA'A)$, so that pairs of corresponding points mutually correspond. Important cases of this will arise hereafter.

*If $ABC$ be a triangle and $\sigma'$ be the perspective of a figure $\sigma$, $A$ and $BC$ being the centre and axis of perspective, and $\sigma''$ be the perspective of $\sigma'$, $B$ and $AC$ being the centre and axis of perspective, and $\sigma'''$ be the perspective of $\sigma''$, $C$ and $AB$ being the centre and axis of perspective, then, provided the anharmonic ratios of the perspective be the same in each case, the figures $\sigma$ and $\sigma'''$ coincide.*

Let $P$ be any point of $\sigma$ and let the anharmonic ratio of the perspective be in each case $\lambda$. Then, if $AP$ meet $BC$ in $K$, the point $P'$ of $\sigma'$ corresponding to $P$ is determined by $(AKPP')=\lambda$; so $P''$ the point of $\sigma''$ corresponding to $P'$ is given by $(BLP'P'')=\lambda$.

$\therefore\ (AKPP')=(BLP'P'')=(LBP''P')$.

Therefore $P$ and $P''$ are collinear with $C$. Also $(CMP''P)=(B\,.\,CMP''P)$
$=(KAP'P)=(AKPP')=\lambda$.



Hence $P$ corresponds to $P''$, when $C$ and $BA$ are the centre and axis of Perspective.

An important particular case of the preceding arises when the three perspectives are harmonic. (See Art. 107.)

**30.** The important theorem of Art. 25—if for the present the use of imaginary points be excluded—requires the two circles to intersect in a pair of real points. From the following theorem it may be ascertained when this will be the case.

*$A$, $A'$, $B$, $B'$ are four collinear points and circles are described on $AA'$ and $BB'$ to contain angles $\alpha$ and $\beta$ respectively.*

(1) *If the points occur in the order $ABA'B'$, the circles will always intersect in real points.*

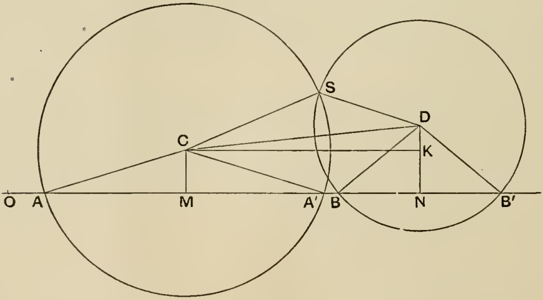
(2) *If the points occur in the order $AA'BB'$, the circles will intersect in real points, if*

$$(BB'AA') > \frac{\cos^2\frac{\alpha-\beta}{2}}{\cos^2\frac{\alpha+\beta}{2}}.$$

(3) *If the points occur in the order $ABB'A'$, the circles will intersect in real points, if*

$$(BB'AA') < \frac{\sin^2\frac{\alpha-\beta}{2}}{\sin^2\frac{\alpha+\beta}{2}}.$$

(1) In this case the theorem is obvious.



(2) Let $C$ and $D$ be the centres of the circles, $M$ and $N$ the feet of the perpendiculars for these points on $AA'BB'$, and $S$ a point of intersection of the circles. Then the condition that $S$ should be real is that

$$(CS+SD)^2 > CD^2 \quad \text{..........................................(i).}$$

Then angle $ACM$ = angle $MCA' = \alpha$,

and angle $BDN$ = angle $NDB' = \beta$.

Let $O$ be any point on the line $AA'BB'$ and $CK$ the perpendicular from $C$ on $DN$. Then (i) becomes

$$(CS+SD)^2>DK^2+CK^2 \qquad \text{(ii)}.$$

But $$CS=\frac{OA'-OA}{2\sin\alpha}, \qquad SD=\frac{OB'-OB}{2\sin\beta},$$

$$CK=\frac{OB'+OB}{2}-\frac{OA'+OA}{2}, \qquad DK=\frac{OB'-OB}{2\tan\beta}-\frac{OA'-OA}{2\tan\alpha}.$$

Hence (ii) becomes

$$\left(\frac{OA'-OA}{2\sin\alpha}+\frac{OB'-OB}{2\sin\beta}\right)^2>\left(\frac{OB'-OB}{2\tan\beta}-\frac{OA'-OA}{2\tan\alpha}\right)^2 + \left(\frac{OB'+OB}{2}-\frac{OA'+OA}{2}\right)^2$$

or $$(OA'-OA)(OB'-OB)\frac{1+\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\sin\beta} >2(OB-OA)(OB'-OA');$$

$$\therefore \quad \frac{1+\cos(\alpha-\beta)}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}>\frac{AB\,.\,A'B'}{AA'\,.\,BB'} \text{ or } > (BA'AB').$$

Hence the required condition is

$$(BB'AA')>\frac{\cos^2\dfrac{\alpha-\beta}{2}}{\cos^2\dfrac{\alpha+\beta}{2}}.$$

(3) As in (2) it may be shown that the condition in this case is that

$$(BA'AB')>\frac{\cos(\alpha-\beta)-1}{\cos(\alpha-\beta)-\cos(\alpha+\beta)}>\frac{\sin^2\dfrac{\alpha-\beta}{2}}{\sin^2\dfrac{\alpha-\beta}{2}-\sin^2\dfrac{\alpha+\beta}{2}}$$

or $$(BB'AA')<\frac{\sin^2\dfrac{\alpha-\beta}{2}}{\sin^2\dfrac{\alpha+\beta}{2}}.$$

## EXAMPLES.

(1) Given in a plane four lines $a$, $b$, $a'$, $b'$, and two points $A$ and $A'$, what (if any) condition must hold if $a$, $b$, $A$ correspond to $a'$, $b'$, $A'$ in plane perspective? Construct the perspective.

$A'$ must be situated on the line joining $a'b'$ to the point, where $s$ the axis of perspective meets $A\,.\,ab$.

(2) Two pairs of straight lines $ab$ and $de$ are given and $c$ is the line joining $ab$ to $de$. On $c$ any point $C$ is taken and through $C$ any two lines are drawn to cut $a$ and $e$ in $A$ and $E$, and $b$ and $d$ in $B$ and $D$.

Show that the points $ae$, $bd$ and $AB\,.\,DE$ are collinear.

Consider the perspective in which

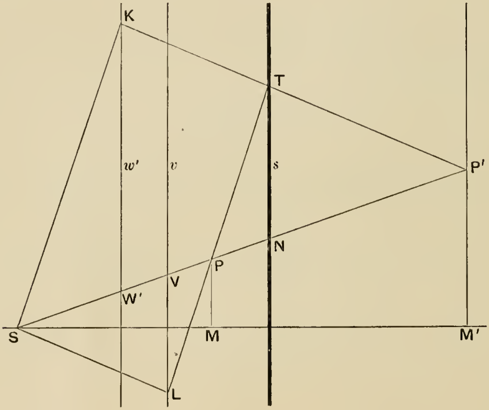
$a$ corresponds to $e$
$b$ „ „ $d$

and $C$ is the centre of perspective.

$A$ corresponds to $E$
$B$ „ „ $D$
$\therefore AB$ „ „ $DE$.

Therefore $ae$, $bd$, and $AB \,.\, DE$ are collinear on $s$ the axis of perspective.

(3) In two figures $\sigma$ and $\sigma'$ in plane perspective prove that the ratio of the distances of two corresponding points $P$ and $P'$ from $S$ is equal (1) to the ratio of the distance of the point $P$ from the vanishing line in its figure, to the distance of the other vanishing line from $S$, and (2) to the ratio of the distances of the two points from the central line.



Let $S$ and $s$ be the centre and axis of perspective. Let $P$ and $P'$ be a pair of corresponding points which must be collinear with $S$. Let $SPP'$ meet $s$ in $N$. Join $P$ and $P'$ to $T$ any point on $s$. Let a line parallel to $PT$ through $S$ meet $TP'$ in $K$ and a line through $S$ parallel to $TP'$ meet $TP$ in $L$. Then $L$ and $K$ are respectively on the vanishing lines $v$ and $w'$, which are parallel to $s$. Let these vanishing lines meet $SPP'$ in $V$ and $W'$ and drop perpendiculars $PM$ and $P'M'$ from $P$ and $P'$ on the central line, *i.e.* on the perpendicular from $S$ on $s$.

Then $\frac{SP}{SP'} = \frac{PM}{P'M'}$ by similar triangles.

Also $$(SPNP') = (S\infty NW'),$$

$$\therefore\ \frac{SW'}{PN} = \frac{SP'}{PP'}, \qquad \therefore\ \frac{SW'}{SW'-PN} = \frac{SP'}{SP'-PP'},$$

$$\therefore\ \frac{SW'}{VP} = \frac{SP'}{SP} \text{ since } SW' = VN.$$

(4) Prove that if the lines joining the corresponding angular points of two triangles in the same plane are concurrent, then the triangles are, in an infinite number of ways, the projections from two points in space of the same triangle lying outside their plane.

(5) Prove that the locus of the vertex from which a system of four fixed points in a plane can be conically projected into a square is a circle in a plane at right angles to the third diagonal of the quadrilateral formed by the four points.

# CHAPTER V

## APPLICATIONS OF PROJECTION AND OF PLANE PERSPECTIVE. TRIANGLES IN PERSPECTIVE

**31.** *If the three sets of points $BCA_1A_2$; $CAB_1B_2$; $ABC_1C_2$ are collinear, the expressions*

$$\frac{BA_1}{CA_1} \cdot \frac{CB_1}{AB_1} \cdot \frac{AC_1}{BC_1} \quad \textit{and} \quad \frac{BA_1 \, . \, BA_2}{CA_1 \, . \, CA_2} \cdot \frac{CB_1 \, . \, CB_2}{AB_1 \, . \, AB_2} \cdot \frac{AC_1 \, . \, AC_2}{BC_1 \, . \, BC_2}$$

*are projective.*

The first expression may be written

$$\left(\frac{BA_1}{CA_1} : \frac{BS}{CS}\right)\left(\frac{CB_1}{AB_1} : \frac{CS}{AS}\right)\left(\frac{AC_1}{BC_1} : \frac{AS}{BS}\right)$$

or, with notation of Art. 10,

$$(BCA_1 \, . \, S)(CAB_1 \, . \, S)(ABC_1 \, . \, S).$$

Each of these expressions is projective by Art. 10, and therefore their product is also projective.

The second expression is projective because it is the product of two expressions of the form of the first expression.

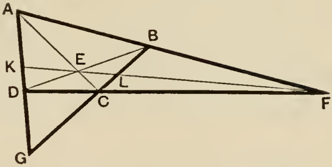
### Projection.

By means of Conical Projection one figure may be deduced from another as explained in the last chapter and the properties of one figure may be inferred from those of the other. From a general theorem a particular case can be deduced or from a particular case a general theorem may be obtained. The theorems of Art. 25 and Art. 27 are those most usually employed. These are as follows, viz. :

(i) From any figure $\sigma$ another figure $\sigma'$ may be obtained by projection, such that (*a*) the angles between any two pairs of lines in $\sigma'$, corresponding to any two given pairs of lines in $\sigma$, may have given values, and (*b*) the line corresponding to any given line in $\sigma$ may be the line at infinity in $\sigma'$.

(ii) From any figure $\sigma$ another figure $\sigma'$ may be obtained by projection, such that any four given points in $\sigma'$ are the projections of any four given points in $\sigma$.

(1) *Any quadrangle can be projected into a square.*



This is a case of (ii). It may be deduced from (i) as follows: Let $ABCD$ be the quadrangle and $E$, $F$, $G$ its diagonal points. Project $GF$ into the line at infinity and the angles $ABC$ and $BEC$ into right angles. Then the resulting figure must be a square.

(2) *The line joining any pair of diagonal points* ($E$ *and* $F$) *of a quadrangle* ($ABCD$) *is cut harmonically by a pair of sides* (*at* $K$ *and* $L$).

Project the quadrangle into a square, $FG$ being projected into the line at infinity. The theorem then reduces to proving that if a line be drawn through the centre of a square parallel to a side, the intercept made on it by the other pair of sides is bisected at the centre.

From this it follows that any side of a quadrangle is divided harmonically at the diagonal point on the side and at the point where the side is met by the opposite side of the diagonal points triangle.

As a particular case it is seen that if through a diagonal point of a quadrangle a straight line be drawn parallel to the opposite side of the diagonal points triangle, the intercept on this line by a pair of opposite sides is bisected at the diagonal point.

(3) *If the corresponding sides of two triangles* $ABC$ *and* $A'B'C'$ *intersect in three collinear points on a straight line* $s$, *then the three lines joining pairs of corresponding vertices are concurrent at a point* $S$. Art. 13 ($a$).

Project the line $s$ into the line at infinity and the theorem reduces to 2 ($a$) of the addendum.

(4) *If the lines joining corresponding vertices of two triangles* $ABC$ *and* $A'B'C'$ *are concurrent at a point* $S$, *then the three pairs of points of intersection of corresponding sides are collinear on a straight line* $s$. Art. 13 ($a$).

Project the point $S$ to infinity, and the theorem reduces to 2 ($b$) of the addendum.

(5) *If a transversal meet the sides of a triangle* $ABC$ *in* $A'B'C'$, *then*

$$\frac{AC'}{BC'}\cdot\frac{BA'}{CA'}\cdot\frac{CB'}{AB'}=1 \quad \text{(Menelaus' Theorem).}$$

The left-hand side of this equation is unaltered by projection. Project the transversal into the line at infinity. In this case $A'$, $B'$, $C'$ are at infinity, and each of the three ratios which compose the expression is equal to unity. Therefore their product is in this, and in all cases, equal to unity.

(6) *If the lines joining the vertices of a triangle* $ABC$ *to any point* $P$ *meet the opposite sides in* $A'$, $B'$, $C'$, *then*

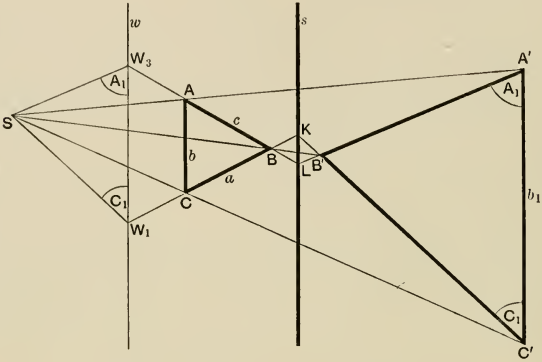
$$\frac{AC'}{BC'}\cdot\frac{BA'}{CA'}\cdot\frac{CB'}{AB'}=-1 \quad \text{(Ceva's Theorem).}$$

Take any triangle and its centroid. Project the triangle $ABC$ into this triangle and the point $P$ into the centroid (Art. 27). Then each of the ratios which compose the expression on the left-hand side becomes equal to $-1$. Therefore the product is $-1$.

### 32. Plane Perspective.

Plane Perspective like Conical Projection may be employed to obtain one figure from another or to deduce one theorem from a related theorem. The following are examples of the construction of figures by Plane Perspective.

(*a*) *To construct the perspective of a given triangle $ABC$, so that it may be equal in dimensions to a given triangle $A_1B_1C_1$.*



Denote the opposite sides of the triangles by the corresponding small letters. Take any line $w$ parallel to $b$ as vanishing line. Let $a$ and $c$ meet $w$ in $W_1$ and $W_3$. Through $W_1$ and $W_3$ draw lines making angles $C_1$ and $A_1$ with $w$, to meet at $S$. Take $S$ as centre of perspective. On $SC$ take $C'$ such that $\frac{SC'}{SC}=\frac{b_1}{b}$. Through $C'$ draw $C'A'$ parallel to $w$ to meet $SA$ in $A'$. Through $A'$ and $C'$ draw lines parallel to $SW_3$ and $SW_1$ to meet $AB$ and $BC$ in $L$ and $K$ and to intersect in $B'$.

Then, in the perspective with $S$ as centre, $w$ as vanishing line and $C$ and $C'$ as corresponding points, $B$ and $B'$ correspond. Hence $ABC$ and $A'B'C'$ are in perspective and consequently $KL$ ($s$) is parallel to $w$. The triangles $A_1B_1C_1$ and $A'B'C'$ are similar by construction. Also since

$$\frac{AC}{A'C'}=\frac{SC}{SC'}=\frac{b}{b_1},$$

the sides $A'C'$ and $A_1C_1$, and therefore the triangles $A_1B_1C_1$ and $A'B'C'$, are equal.

(*b*) *To construct the perspective of a given quadrangle $ABCD$ so that it may be*

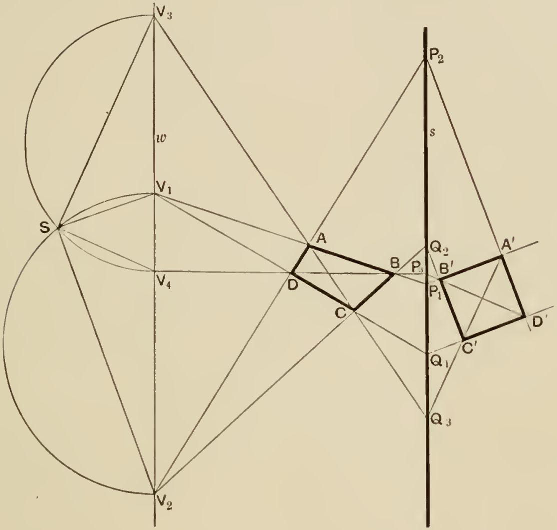
(i) *a square.*

Let $V_1$ and $V_2$ be a pair of diagonal points of the quadrangle. Take $V_1V_2$ as vanishing line and let the third pair of sides of the quadrangle meet $V_1V_2$ in $V_3$ and $V_4$. On $V_1V_2$ and on $V_3V_4$, as diameters, describe two circles which will intersect at some point $S$. Take $S$ as centre of perspective and some line $s$ parallel to $w$ as axis. The perspective of the quadrangle will then be a square.

For since $V_1$ and $V_2$ are on the vanishing line, the resulting figure is a parallelogram.

Since $SV_1$ and $SV_2$ are at right angles and are parallel to the sides, it is a rectangle.

Since $SV_3$ and $SV_4$ are at right angles and are parallel to the diagonals, it is a square.



(ii) *a square of given dimensions.*

The dimensions of the square in (i) vary directly as the distance between $w$ and $s$ (Art. 24). Let $d$ be the length of the side of the given square. On $V_2Q_2$ take a point $Q_2'$ such that

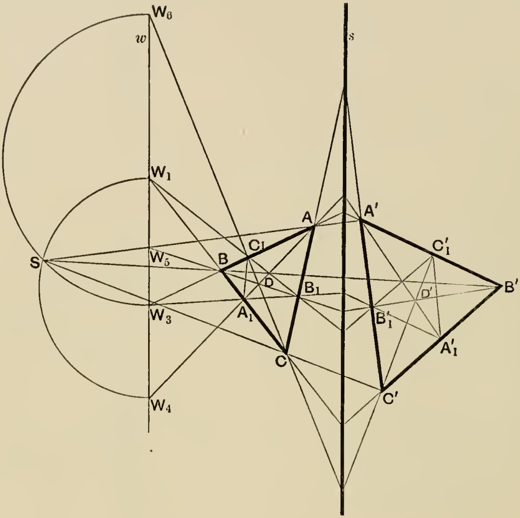
$$\frac{V_2Q_2'}{V_2Q_2}=\frac{d}{B'C'}.$$

If a line parallel to $w$ through $Q_2'$ be taken as axis of perspective, the resulting figure will be a square of side $d$.

(iii) *an equilateral triangle and its orthocentre.*

Let $ABCD$ be the quadrangle and let $AD$, $BD$, $CD$ meet $BC$, $CA$, $AB$ respectively in $A_1B_1C_1$. Let $BC$ meet $B_1C_1$ in $W_1$ and $AB$ meet $A_1B_1$ in $W_3$. Then the ranges $ABC_1W_3$ and $CBA_1W_1$ are harmonic (Art. 31 (2)). Take $W_1W_3$ as vanishing line ($w$). Let $AD$ and $CD$ meet $w$ in $W_4$ and $W_6$ respectively. On $W_1W_4$ and on $W_3W_6$ as diameters describe circles intersecting in $S$. Take $S$ as centre of perspective and any line parallel to $w$ as axis of perspective.

In the perspective figure the points corresponding to $A_1B_1C_1$ will bisect the sides of the triangle corresponding to $ABC$ and the lines corresponding to $AD$, $BD$, $CD$ will be perpendicular to those corresponding to $BC$, $CA$, $AB$. Hence the figure meets the requirements of the case. Its dimensions depend on the position of $s$.



From the above it is seen that if two of the three segments determined on a line by any four points subtend a right angle at a given point, the third segment also subtends a right angle at that point.

(iv) *a quadrangle of given dimensions.* See Art. 65, Chapter IX.

### 33. Particular cases of Triangles in Perspective.

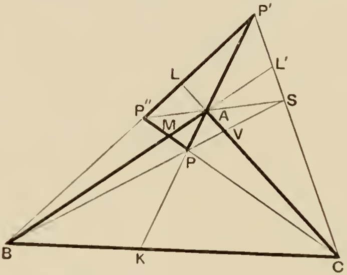
(i) *If three coplanar triangles are two by two in perspective, and have the same axis of perspective, their three centres of perspective are collinear.*

Let the triangles be (1) $ABC$, (2) $A'B'C'$ and (3) $A''B''C''$, and let their opposite sides be denoted by the corresponding small letters. Since the triangles are two by two in perspective with the same axis of perspective, the sides $a$, $a'$, $a''$ are concurrent. Therefore the triangles $BB'B''$ and $CC'C''$ are in perspective. Therefore the points $BB' . CC'$; $B'B'' . C'C''$; $B''B . C''C$ are collinear. But these points are the centres of perspective of the triangles (1), (2) and (3) taken two by two.

(ii) *If three coplanar triangles are two by two in perspective and have the same centre of perspective, their three axes of perspective are concurrent.*

Let the triangles be (1) $abc$, (2) $a'b'c'$, and (3) $a''b''c''$, and let their opposite vertices be denoted by the corresponding large letters. Since the triangles are two by two in perspective with the same centre of perspective, the vertices $A$, $A'$, $A''$ are collinear. Therefore the triangles $bb'b''$ and $cc'c''$ are in perspective. Therefore the lines $bb'.cc'$ ; $b'b''.c'c''$ ; $b''b.c''c$ are concurrent. But these lines are the axes of perspective of the triangles (1), (2) and (3) taken in pairs.

(iii) *If a triangle be circumscribed to another triangle and is in perspective with it, the three vertices of the former triangle may be deduced from each other as harmonic perspectives with respect to the vertices and opposite sides of the latter triangle.*



Let the triangle $PP'P''$ be circumscribed to the triangle $ABC$ and in perspective with it, as in the figure, $S$ being the centre of perspective. Let $AP$, $BP'$ and $CP''$ meet the opposite sides of $ABC$ in $K$, $L$ and $M$. Then from the quadrangle $PP''P'S$,

$$(BVPS) = -1.$$

But $$(KAPP') = (BVPS) = -1.$$

Similarly the ranges $(BLP'P'') = -1$ and $(CMP''P) = -1$.

*Conversely: If $P$, $P'$, $P''$ are harmonic perspectives of each other with respect to the vertices $A$, $B$, $C$, and the opposite sides of a triangle $ABC$, the triangles $PP'P''$ and $ABC$ are in perspective.*

Take the perspective of $P$ with centre $B$ and axis $AC$; this will be a point $S$ such that $(BVPS) = -1$. Then (Art. 29) $S$ is the perspective of $P'$ with regard to $C$ and $AB$. To show that $P''A$ passes through $S$ it is necessary to prove that $L'B$, $LC$ and $P''S$ are concurrent.

This is the case since $(P'P''BL) = -1 = (P'SL'C)$.

The correlative theorems are :

*If a triangle be inscribed in another triangle and in perspective with it, the three sides of the former triangle may be deduced from each other as harmonic perspectives with respect to the vertices and opposite sides of the latter triangle.*

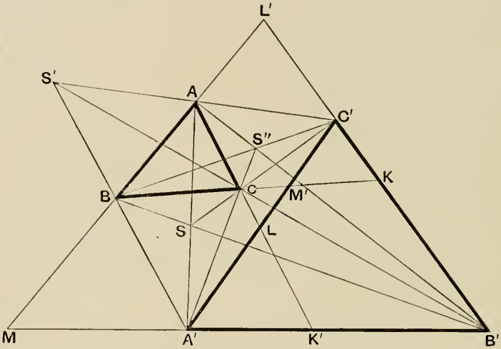
*Conversely: If $p$, $p'$, $p''$ are harmonic perspectives with respect to the vertices $A$, $B$, $C$, and the opposite sides of a triangle $abc$, the triangles $pp'p''$ and $abc$ are in perspective.*

(iv) *If two coplanar triangles are copolar in two different ways, such that each pair of corresponding vertices in the one case are non-corresponding vertices in the other, they are also copolar in a third way and the three poles are the vertices of a triangle copolar in three ways to either of the former triangles.*

Let $ABC$ and $A'B'C'$ be the triangles and denote the opposite sides by the corresponding small letters. In the figure, the triangles are in perspective with the centres of perspective at $S$ and $S'$. It is required to prove that $A'C$, $AB'$ and $BC'$ are concurrent at some point $S''$.

Because the triangles are in perspective with $S$ as centre of perspective, $K(a.a')$, $L(b.b')$, $M(c.c')$ are collinear. Also in the perspective centre $S'$

$K(a.a')$ corresponds to $K'(c'.b)$,
$L(b.b')$ „ „ $L'(a'.c)$,
$M(c.c')$ „ „ $M'(b'.a)$.



Hence, since $K$, $L$, $M$ are collinear, $K'$, $L'$, $M'$ are collinear. Therefore the triangles are in perspective with $c'$, $a'$, $b'$ corresponding to $b$, $c$, $a$ respectively and the lines $BC'$, $CA'$, and $AB'$ are concurrent at some point $S''$.

The triangle $ABC$ is obviously in perspective with the triangle $SS'S''$ from $A'$, $B'$ and $C'$.

## EXAMPLES.

(1) Given five points, no three of which are collinear, project them into five points such that one may be the centroid and the other the orthocentre of the remaining three.

(2) Through the points of intersection of the diagonals of a quadrilateral straight lines are drawn parallel to the four sides to meet the sides which are respectively opposite to those to which they are drawn parallel. Prove that the four points of intersection are collinear.

(3) Show that if two coplanar triangles are in perspective and a vertex of one lies on a non-corresponding side of the other (*e.g.* $A'$ on $AB$), then provided the centre of perspective is not at $B$, a pair of corresponding sides must coincide; examine the position of the centre and axis of perspective.

(4) If two quadrangles are in perspective in three ways with the vertices and opposite sides of a triangle as centres and axes of perspective, the vertices are collinear in pairs with a fourth point.

(5) If $A_1B_1C_1$ and $A_2B_2C_2$ be two triangles in perspective, the triangle $B_1C_2 . B_2C_1$, $C_1A_2 . C_2A_1$, $A_1B_2 . A_2B_1$ is in perspective with $A_1B_1C_1$ and $A_2B_2C_2$ and the three centres of perspective lie on a straight line.

(6) If on each of three concurrent straight lines $a$, $b$, $c$ three points are taken, these form the vertices of 36 sets of triangles in perspective in pairs, and the axes of perspective pass three by three through 36 points which lie four by four on 27 straight lines.

(7) If four triangles (1) $A_1B_1C_1$, (2) $A_2B_2C_2$, (3) $A_3B_3C_3$, (4) $A_4B_4C_4$ are such that (1) and (2), (2) and (3), (3) and (4), (4) and (1) are in perspective with $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ for corresponding vertices, and $S_1S_2S_3S_4$ for centres of perspective, and $s_1s_2s_3s_4$ for axes, prove that, if $S_1S_2S_3S_4$ are collinear, $s_1s_2s_3s_4$ are concurrent.

# CHAPTER VI

## ANHARMONIC FORMS:—RANGES AND PENCILS IN PERSPECTIVE. RANGES AND PENCILS PROJECTIVE. RANGES AND PENCILS SUPERPOSED

### 34. Self-corresponding elements.

If two projective ranges are constructed on the same base, they are termed *superposed projective ranges.* Any point upon the base may be looked upon as an element of either range and to a given point, considered as an element of one range, there generally corresponds some other point on the base regarded as an element of the other range.

In certain cases a pair of corresponding elements coincide in one point and such a point is termed *a self-corresponding point* of the given projective ranges.

A pair of projective ranges on different bases may also have a self-corresponding point at the point of intersection of the bases.

If two projective pencils are constructed with the same vertex, they are termed *superposed projective pencils.* Any line through the vertex may be looked upon as an element of either pencil and to a given line, considered as an element of one pencil, there generally corresponds some other line through the vertex regarded as an element of the other pencil.

In certain cases a pair of corresponding elements coincide in one line and such a line is termed *a self-corresponding ray* of the given projective pencils.

A pair of projective pencils with different vertices may also have a self-corresponding ray in the line joining their two vertices.

### Ranges and pencils in perspective.

Two ranges are said to be in perspective when the lines joining pairs of corresponding points pass through a fixed point.

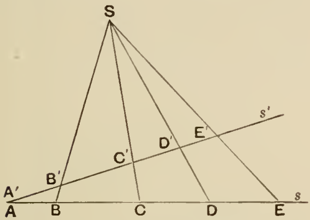
Two pencils are said to be in perspective when the points of intersection of pairs of corresponding rays lie on a fixed line.

*If in two projective ranges*

(1) *the connectors of three pairs of corresponding points are concurrent,*

or (2) *the point of intersection of the bases is a self-corresponding point,*

*the two ranges are in perspective.*



Let the projective ranges $A$, $B$, $C$, $D$, $E$, ... and $A'$, $B'$, $C'$, $D'$, $E'$, ... be situated on bases $s$ and $s'$.

(1) Let $BB'$, $CC'$, $DD'$ be concurrent at the point $S$.

Join $S$ to $E$ any point on $s$ to meet $s'$ at $E_1$.

Then $(BCDE) = (B'C'D'E_1)$,

but $(BCDE) = (B'C'D'E')$

since the ranges are projective. Therefore $E_1$ and $E'$ coincide and every pair of corresponding elements are collinear with $S$.

(2) Let $ss'$ be a self-corresponding point, viz. $A$ or $A'$, and let $BB'$, $CC'$ meet at $S$.

Join $S$ to $E$ any point on $s$ to meet $s'$ at $E_1$.

Then $(ABCE) = (A'B'C'E_1)$,

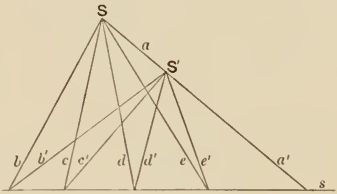
but $(ABCE) = (A'B\,C'E')$

*If in two projective pencils*

(1) *the points of intersection of three pairs of corresponding rays are collinear,*

or (2) *the connector of the vertices is a self-corresponding ray,*

*the two pencils are in perspective.*



Let the projective pencils $a$, $b$, $c$, $d$, $e$, ... and $a'$, $b'$, $c'$, $d'$, $e'$, ... have for their vertices $S$ and $S'$.

(1) Let $bb'$, $cc'$, $dd'$ be collinear on the line $s$.

Join the intersection of $s$ with $e$, any line through $S$, to $S'$ by a line $e_1$.

Then $(bcde) = (b'c'd'e_1)$,

but $(bcde) = (b'c'd'e')$

since the pencils are projective. Therefore $e_1$ and $e'$ coincide and every pair of corresponding elements intersect on $s$.

(2) Let $SS'$ be a self-corresponding ray, viz. $a$ or $a'$, and let $bb'$, $cc'$ be the line $s$.

Join the point of intersection of $s$ with $e$, any line through $S$, to $S'$ by a line $e_1$.

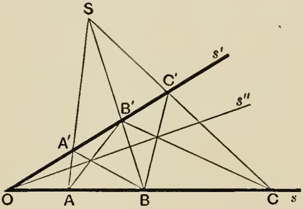
Then $(abce) = (a'b'c'e_1)$,

but $(abce) = (a'b'c'e')$

since the ranges are projective. Therefore as before $E_1$ and $E'$ coincide.

From the above it follows that two pairs of corresponding elements completely determine two ranges in perspective.

*If two ranges are in perspective the transverse connectors of any two pairs of corresponding points intersect on a fixed straight line.*



Let $A, B, C, \ldots$ and $A', B', C', \ldots$ be the ranges, in which $AA'$, $BB'$, $CC'$, ... are collinear with a fixed point $S$. Consider the triangles $A'BC'$ and $AB'C$. They are in perspective with $S$ for centre of perspective. Therefore $AB'.BA'$ and $BC'.CB'$ lie on a line $s''$ which passes through $O$ the point of intersection of the bases.

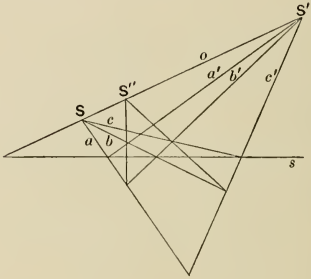
If $AA'$, $BB'$ are regarded as two pairs of fixed corresponding points, which determine the ranges, the line $s''$ is fixed and the variable point $BC'.CB'$ lies on this line.

This result also follows from the properties of Harmonic Perspective when $S$ is the centre and $s''$ the axis of Perspective.

since the ranges are projective. Therefore as before $e_1$ and $e'$ coincide.

From the above it follows that two pairs of corresponding rays completely determine two pencils in perspective.

*If two pencils are in perspective the points of intersection of pairs of the non-corresponding rays are collinear with a fixed point.*



Let $a, b, c, \ldots$ and $a', b', c', \ldots$ be the ranges, in which $aa'$, $bb'$, $cc'$, ... lie on a fixed line $s$. Consider the triangles $a'bc'$ and $ab'c$. They are in perspective with $s$ for axis of perspective. Therefore $ab'.ba'$, $bc'.cb'$ intersect in a point $S''$ which lies on $o$ the connector of the vertices.

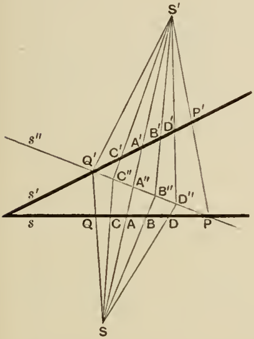
If $aa'$, $bb'$ are regarded as two pairs of fixed corresponding rays, which determine the pencils, the point $S''$ is fixed and the variable line $bc'.cb'$ passes through this point.

This result also follows from the properties of Harmonic Perspective when $s$ is the axis and $S''$ the centre of Perspective.

## 35. Projective ranges and pencils.

*Construction of corresponding elements of projective forms*.*

Given three pairs of corresponding points $A$ and $A'$, $B$ and $B'$, $C$ and $C'$ on two bases $s$ and $s'$, to construct the projective ranges determined by these corresponding points.



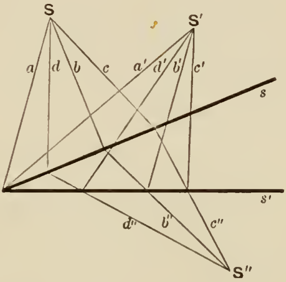
Take $S$ and $S'$ any two points on $AA'$. Construct $s''$ the line joining $SC . S'C'$ to $SB . S'B'$. Then the pencils $S . ABC \ldots$

and $S' . A'B'C' \ldots$

have the self-corresponding ray $SS'$, and are therefore in perspective, with $s''$ as axis of perspective (Art. 34).

To construct the point corresponding to any point $D$ on $s$, join $SD$ to meet $s''$ in $D''$. Join $D''S'$ to meet $s'$ in $D'$. Then $D'$ is the point corresponding to $D$,

Given three pairs of corresponding rays $a$ and $a'$, $b$ and $b'$, $c$ and $c'$ passing through two vertices $S$ and $S'$, to construct the projective pencils determined by these corresponding rays.



Take $s$ and $s'$ any two lines through $aa'$. Construct $S''$ the point of intersection of $sc . s'c'$ and $sb . s'b'$. Then the ranges $s . abc \ldots$

and $s' . a'b'c' \ldots$

have the self-corresponding point $ss'$, and are therefore in perspective, with $S''$ as centre of perspective (Art. 34).

To construct the ray corresponding to any ray $d$ through $S$, join $sd$ to $S''$ by $d''$. Join $d''s'$ to $S'$ by $d'$. Then $d'$ is the ray corresponding to $d$, for the same for the same projections which determine $A'B'C'$ from $ABC$ also determine $D'$ from $D$.

---

* For construction of self-corresponding elements of two superposed ranges see Art. 109.

If $s''$ meet $s$ and $s'$ in $P$ and $Q'$, $P'$ and $Q$, the points corresponding to $P$ and $Q'$, are found as the intersects of $S'P$ and $SQ'$ with $s'$ and $s$ respectively.

Any line may be taken as the line $s''$. For, if $s''$ meet $s$ and $s'$ in $P$ and $Q'$, and $P'$ and $Q$ be the corresponding points, then $S$ and $S'$ are found as the points of intersection of $QQ'$ and $PP'$ with the line joining any pair of corresponding points $A$ and $A'$.

projections which determine $a'b'c'$ from $abc$ also determine $d'$ from $d$.

If the lines joining $S''$ to $S$ and $S'$ be $p$ and $q'$, $p'$ and $q$, the rays corresponding to $p$ and $q'$, are found as the connectors of $s'p$ and $sq'$ to $S'$ and $S$ respectively.

Any point may be taken as the point $S''$. For, if the lines joining $S''$ to $S$ and $S'$ be $p$ and $q'$, and $p'$ and $q$ be the corresponding rays, then $s$ and $s'$ are found as the lines joining the points $qq'$ and $pp'$ to the point of intersection of any pair of corresponding rays $a$ and $a'$.

From the preceding it follows that

(1) *A range can always be constructed with which each of two given projective ranges is in perspective.*

(2) *Three pairs of corresponding points determine two projective ranges.*

(3) *Two ranges, such that the anharmonic ratio of any four points of the one is equal to the anharmonic ratio of the four corresponding points of the other, are projective.*

Theorem (3) of the above may be proved by taking $A$, $B$, $C$ and $A'$, $B'$, $C'$ to determine the ranges, and $D$ and $D'$ for corresponding points. The point $D_1$ determined by the above construction as corresponding to $D$, is such that

$$(A'B'C'D_1) = (ABCD)$$

and therefore coincides with $D'$.

*A pencil can always be constructed with which each of two given projective pencils is in perspective.*

*Three pairs of corresponding rays determine two projective pencils.*

*Two pencils, such that the anharmonic ratio of any four rays of the one is equal to the anharmonic ratio of the four corresponding rays of the other, are projective.*

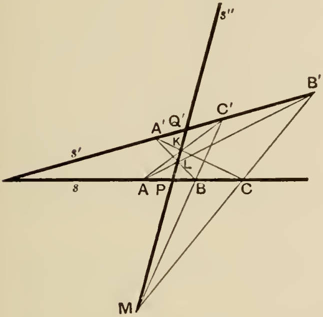
Theorem (3) of the above may be proved by taking $a$, $b$, $c$ and $a'$, $b'$, $c'$ to determine the pencils, and $d$ and $d'$ for corresponding rays. The ray $d_1$ determined by the above construction as corresponding to $d$, is such that

$$(a'b'c'd_1) = (abcd)$$

and therefore coincides with $d'$.

### 36. Particular case of the preceding.

Take $S$ at any point $A'$ of the range on $s'$, and $S'$ at the corresponding point $A$ on $s$.



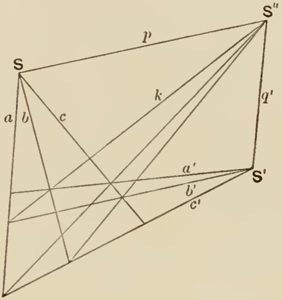
Let $s''$ meet $s$ and $s'$, as before, in $P$ and $Q'$.

Then, by construction, $P$ corresponds to the point $ss'$ regarded as a point of the range on $s'$, and $Q'$ corresponds to the point $s's$ regarded as a point of the range on $s$.

Hence $Q'$ and $P$ are fixed points and $s''$ is therefore a fixed line, as long as $S$ and $S'$ are taken at pairs of corresponding points of the ranges. But $A$, $A'$ and $B$, $B'$ are any two pairs of corresponding points and $AB'$ and $A'B$ intersect in $s''$.

It follows therefore that the lines joining transversely any two pairs of corresponding points of two projective ranges intersect on a fixed line.

Take $s$ as the ray $a'$ of the pencil vertex $S'$, and $s'$ as the corresponding ray $a$ of the pencil vertex $S$.



Let the lines joining $S$ and $S'$ to $S''$ be as before $p$ and $q'$.

Then, by construction, $p$ corresponds to line $SS'$ regarded as ray of pencil $S'$, and $q'$ corresponds to line $S'S$ regarded as ray of pencil $S$.

Hence $q'$ and $p$ are fixed lines and $S''$ is therefore a fixed point, as long as $s$ and $s'$ are a pair of corresponding rays of the pencils. But $a$, $a'$ and $b$, $b'$ are any two pairs of corresponding rays and $ab'$ and $a'b$ are collinear with $S''$.

It follows therefore that the points of intersection of the non-corresponding rays of any two pairs of corresponding rays of two projective pencils are collinear with a fixed point.

If any three pairs of points $AA'$, $BB'$, $CC'$ be taken, these determine two projective ranges and the points

$$AB'.BA';\ BC'.CB';\ CA'.AC'$$

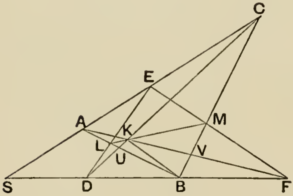
are collinear.

From this there follows

**Pascal's theorem for a Pair of Lines, viz.:**

*If the alternate vertices of a hexagon lie three by three on two straight lines, then the three points of intersection of the pairs of opposite sides are collinear.*

This theorem may also be proved as follows:



Let $ABCDEF$ be a hexagon, such that $A$, $E$, $C$ and $D$, $B$, $F$ lie on two straight lines which meet at $S$. Let $K$, $L$, $M$ be the points of intersection of pairs of opposite sides, and let $U$ and $V$ be the points $AB.DC$ and $AF.CB$. Join $KL$ and $KM$. The condition that $K$, $L$, $M$ should be collinear is

$$(K.LAUB)=(K.MVCB),$$

or $$(LAUB)=(MVCB),$$

or $$(D.EACS)=(F.EACS),$$

which is true.

If any three pairs of rays $aa'$, $bb'$, $cc'$ be taken, these determine two projective ranges and the lines

$$ab'.ba';\ bc'.cb';\ ca'.ac'$$

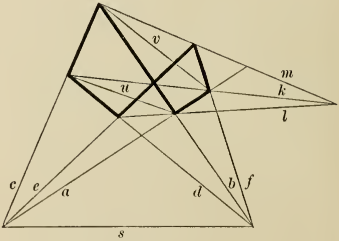
are concurrent.

From this there follows

**Brianchon's theorem for a Pair of Points, viz.:**

*If the alternate sides of a hexagon pass three by three through two fixed points, then the three lines joining the pairs of opposite vertices are concurrent.*

This theorem may also be proved as follows:



Let $abcdef$ be a hexagon, such that $a$, $e$, $c$ and $d$, $b$, $f$ meet at two points the connector of which is $s$. Let $k$, $l$, $m$ be the connectors of pairs of opposite vertices, and let $u$ and $v$ be the lines $ab.dc$ and $af.cb$. The condition for $k$, $l$, $m$ to be concurrent is

$$(k.laub)=(k.mvcb),$$

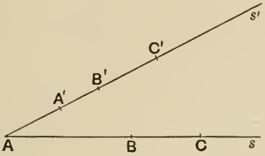
or $$(laub)=(mvcb),$$

or $$(d.eacs)=(f.eacs),$$

which is true.

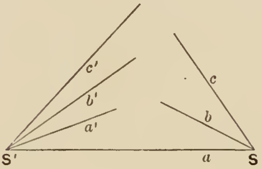
### 37. Connexion between Projective Pencils and Ranges and those in Plane Perspective.

*Two projective ranges may be placed in plane perspective by moving either range along its base.*



Let $A'$ be the point of one range which corresponds to the point of intersection of the bases looked upon as a point of the other. If the points $A'$, $B'$, $C'$ are moved through a distance $AA'$ along $s'$ the ranges will have a self-corresponding point at $A$ and be in perspective with the point $BB'.CC'$ (in the new position) for centre of perspective.

*Two projective pencils may be placed in plane perspective by rotating either pencil round its vertex.*



Let $a'$ be the ray of one pencil which corresponds to the line joining the vertices of the pencils looked upon as a ray of the other. If the rays $a'$, $b'$, $c'$ are rotated round $S'$ through an angle $\widehat{aa'}$ the pencils will have a self-corresponding ray in $a$ and be in perspective with the line $bb'.cc'$ (in the new position) for axis of perspective.

In the preceding (right-hand side) the rotation may be performed in either of two ways:

(1) the ray $a'$ may be rotated through the acute angle $a'S'a$, or

(2) through the angle $\pi - a'S'a$.

The axis of perspective is the same in both cases.

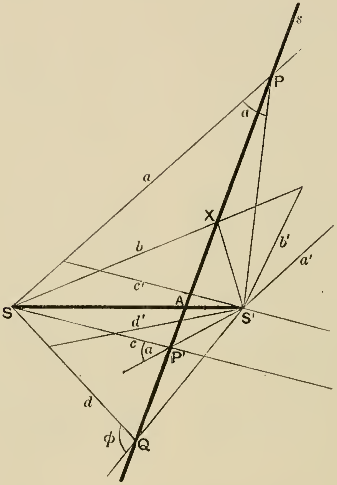
Conversely, *any two projective pencils may be obtained by the rotation round its vertex of either of some two pencils in plane perspective.*

**38.** *Two projective pencils may have two pairs, one pair or no pair of parallel corresponding rays.*

Rotate one of the pencils round its vertex $S'$ till the pencils are in plane perspective and let $s$ be the axis of perspective in the displaced position.

(1) Let the axis of perspective $s$ meet $SS'$ between $S$ and $S'$ at $A$.

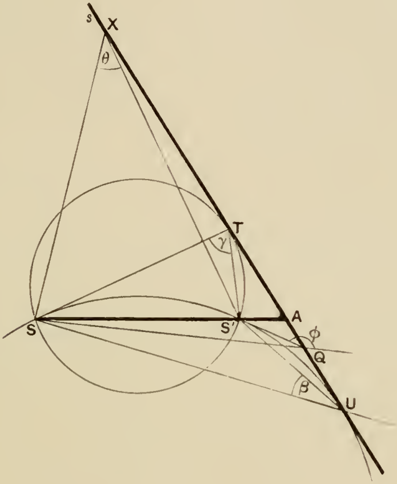
Pairs of corresponding rays of the pencils in the displaced position are obtained by joining a variable point $X$ on $s$ to $S$ and $S'$. When the point $X$ is very distant from $A$ in the upper part of the figure the angle $\theta$ between corresponding rays is zero. As $X$ moves towards $A$ this angle increases till at $A$ it is $\pi$. After passing $A$ the external angle, marked $\phi$ at $Q$, increases from 0 to $\pi$, which value it reaches when $X$ is taken at an infinite distance below $A$. Hence the internal angle between the rays above $SS'$ and the external angle below $SS'$ each have any given value $\alpha$ ($< \pi$). Therefore, when the pencil vertex $S'$ is rotated back to its original position round $S'$, through an angle $\alpha$, there will be two pairs of parallel corresponding rays. These correspond to the two pairs of rays passing through $P$ and $P'$ which originally contained an angle $\alpha$.



(2) Let the axis of perspective $s$ of the pencils meet $SS'$ at $A$, where $A$ is not situated between $S$ and $S'$.

In this case when $X$ is taken at infinity at the top of the figure the angle $\theta$ between the lines joining $X$ to $S$ and $S'$ is zero. This angle $\theta$

increases from zero till $P$ is at $T$ in which case $STS'$ is a maximum ($\gamma$). The point $T$ is determined as the point of contact of a circle through $S$ and $S'$ which touches the axis of perspective. The angle $\theta$ then decreases till at $A$, where $SS'$ meets the axis of perspective, it is again zero.



Hence for a positive rotation of $S'$ through an angle between $o$ and $\gamma$ there are two pairs of parallel corresponding rays.

Below $A$ the angle to be considered is the external angle $\phi$ between $SQ'$ and $SQ$ (produced).

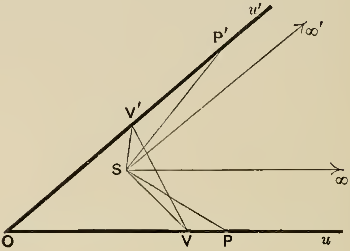
This angle $\phi$ is initially $\pi$ and decreases till at $U$ it is $\pi - \beta$, where $U$ is again the point of contact of a circle through $S$ and $S'$ which touches the axis of perspective. From $U$ to the lower point at infinity on the axis $\phi$ increases from $\pi - \beta$ to $\pi$.

Hence, for a positive rotation between $\pi - \beta$ and $\pi$ of the pencil round $S'$ there are again two pairs of parallel corresponding rays.

For a rotation of the pencil $S'$ in a positive direction of $\gamma$ or $\pi - \beta$ there is one pair of parallel corresponding rays.

For a rotation of the pencil $S'$ in a positive direction between $\gamma$ and $\pi - \beta$ there is no pair of parallel corresponding rays.

**39.** (*a*) **There are two points at which pairs of corresponding points of two projective ranges on different bases subtend equal angles.**



Let the ranges be situated on the bases $u$ and $u'$ which intersect at $O$, and let them be determined (Art. 35) by three pairs of corresponding points $V$, $P$, $\infty$ and $\infty'$, $P'$, $V'$.

Every pair of corresponding points will subtend equal angles at $S$, if $S$ is such that

$$V'SP' = \infty SP$$

and $$P'S\infty' = PSV.$$

In this case the angle $OVS = \infty SV = V'S\infty' = OV'S$. Hence the lines joining $V$ and $V'$ to $S$, make equal angles with $u$ and $u'$ respectively.

If the second pencil is rotated round $S$ through such an angle that $SP$ and $SP'$ lie on each other, the pencils will coincide, and the two ranges will be the intercepts made on two different transversals $u$ and $u'$, by the same pencil.

But it was shown (Art. 10) that in this case

$$VP \,.\, V'P' = \text{constant} = VS \,.\, V'S.$$

Hence the product $VS \,.\, V'S$ is a known quantity.

Hence $S$ is found as the vertex of a triangle described on $VV'$ as base, having the difference of the base angles equal to the difference of the angles $OV'V$ and $OVV'$, and having the product of its sides equal to a given quantity.

Such a triangle can be constructed (Addendum 4) on each side of $VV'$.

That any two pairs of corresponding points subtend equal angles at $S$, determined as above, may be proved as follows :

Consider the triangles $VPS$ and $V'SP'$.

$$V'P' \,.\, VP = V'S \,.\, VS,$$

$$\therefore \ \frac{V'P'}{V'S} = \frac{VS}{VP},$$

also the angles $SV'P$ and $SVP$ are equal.

Therefore the triangles $V'SP'$ and $VPS$ are similar and the angle $V'SP'$ = the angle $VPS$.

Take two other corresponding points $Q$ and $Q'$.

Then $$V'SQ' = VQS,$$

and $$P'SQ' = VQS - VPS = PSQ.$$

(*b*) *There are generally two straight lines on which pairs of corresponding rays of two projective pencils with different vertices determine equal segments.*

There are generally two pairs of parallel corresponding rays of two projective pencils (Art. 38). Draw any transversal parallel to the direction of one of these pairs of parallel corresponding rays to meet the pencils in $A$, $B$, $C$, $\infty$ and $A'$, $B'$, $C'$, $\infty$ respectively, where $\infty$ is the point at infinity on the transversal. Then

$$(ABC\infty) = (A'B'C'\infty) \text{ or } \frac{AC}{BC} = \frac{A'C'}{B'C'}.$$

Hence, if $AC$ and $A'C'$ be equal, $BC$ and $B'C'$ will also be equal. But (Addendum 3) a straight line may be drawn in a given direction so that the intercepts on it made by two given pairs of lines are equal. The line so drawn is one on which three pairs of corresponding rays of the pencils determine equal segments, and on which therefore all pairs of corresponding rays determine equal segments. A second line, parallel to the other pair of parallel corresponding rays of the pencils, may be obtained in a similar manner.

From (*a*) it follows that *any pair of projective ranges may be obtained by the section of the same pencil by a pair of transversals, one transversal being afterwards rotated round the vertex of the pencil.*

From (*b*) it follows that *generally any pair of projective pencils may be obtained by projecting the same range from two different vertices, one pencil being afterwards moved parallel to the base on which the range is situated.*

## 40. Summary of the Properties of two Projective Ranges.

In the case of two projective ranges in which $A$, $B$, $C$ ... correspond to $A'$, $B'$, $C'$ ...

(*a*) There is a vanishing point ($W$ or $V'$) on each base such that

$WA \,.\, V'A' = WB \,.\, V'B' = \ldots$ constant. (Arts. 10 and 39.)

(*b*) Corresponding points may be obtained by projecting on the bases any point on the line joining $SA \,.\, S'A'$ to $SC \,.\, S'C'$ where $S$ and $S'$, the centres of projection, are on the connector of a pair of corresponding points. (Art. 35.)

(*c*) Corresponding points may also be obtained by rotating a given angle round either of two fixed points. (Art. 39.)

(*d*) The ranges are in plane perspective, if the point of intersection of the bases corresponds to itself. (Art. 34.)

(*e*) The ranges may be placed in plane perspective by moving one of them, so that the point of intersection of the bases is a self-corresponding point. (Art. 37.)

### 41. Superposed Projective Ranges and Pencils.

*Two superposed projective ranges or pencils may have one or two, but not more than two, self-corresponding points*.*

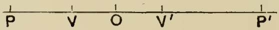
Let the projective pencils $abc \ldots$ and $a'b'c' \ldots$ have a common vertex $S$. Through any point $S'$ in their plane draw lines $a_1b_1c_1 \ldots$ parallel to $a'b'c' \ldots$. Then the pencils $abc \ldots$ and $a_1b_1c_1 \ldots$ are projective. Hence (Art. 38) they have two pairs, one pair or no pair of parallel corresponding rays. If they have two pairs of parallel corresponding rays $e, e_1$ and $f; f_1$, the rays $e'$ and $f'$ of the pencil $a'b'c' \ldots$ which correspond to $e_1$ and $f_1$ must coincide with the rays $e$ and $f$ of the pencil $abc \ldots$. Hence the two given pencils have two self-corresponding rays.

Similarly, if the pencils $abc \ldots$ and $a_1b_1c_1 \ldots$ have one pair or no pair of parallel corresponding rays, the pencils $abc \ldots$ and $a'b'c' \ldots$ have one or no self-corresponding ray. It is obvious that two pencils cannot have more than two self-corresponding rays without entirely coinciding.

If the points of two superposed projective ranges are joined to any vertex two superposed pencils are obtained, and, corresponding to the self-corresponding rays of the pencils, there may be two self-corresponding points, one self-corresponding point, or no self-corresponding point of the ranges.

### 42. Maximum and Minimum Segments of Superposed Projective Ranges.

Let $V$ and $V'$ be the vanishing points of two superposed projective ranges. Then if $P$ and $P'$ are a pair of corresponding points $VP \,.\, V'P' =$ a constant. (Arts. 10 and 39.)



Suppose this constant to be $-K^2$ so that $VP \,.\, V'P' = -K^2$. Then

$$VP(V'P+PP') = -K^2,$$

$$\therefore\ PP' = -\frac{K^2}{VP} - V'P = -\frac{K^2}{VP} - (V'V+VP),$$

$$\therefore\ PP' = \{+2K - V'V\} - \frac{\{K+VP\}^2}{VP} \quad \text{..............................(i)},$$

or

$$PP' = \{-2K - V'V\} - \frac{\{K-VP\}^2}{VP} \quad \text{..............................(ii)}.$$

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* For the construction of self-corresponding elements of two superposed projective ranges see Art. 109.

The least value of the numerator of the second expression is in each case zero.

*Consider* (i).

If $2K - V'V$ is positive $PP'$ is a minimum when

$$VP = -K.$$

If $2K - V'V$ is negative $PP'$ is (numerically) a maximum when

$$VP = -K.$$

*Consider* (ii).

If $-2K - V'V$ is positive $PP'$ is a maximum when

$$VP = +K.$$

If $-2K - V'V$ is negative $PP'$ is (numerically) a minimum when

$$VP = +K.$$

Hence for the values $VP = \pm K$ the segments $PP'$ are one a maximum and the other a minimum. These are termed the maximum and minimum segments. They will be denoted by $EE'$ and $FF'$.

If $VP\,.\,V'P' = +K^2$ the maximum and minimum segments are imaginary.

The self-corresponding points ($K$ and $L$) are given the relation

$$VP\,.\,V'P = -K^2.$$

They are therefore situated between $V$ and $V'$ and are equally distant from these points.

It will be seen from the preceding that if $O$ be the middle point of $VV'$ then

$$EE' = -2(K + OV), \qquad FF' = +2(K - OV),$$

$$OE = K + OV, \qquad OE' = -(K + OV),$$

$$OF = -K + OV, \qquad OF' = (K - OV),$$

$$OK^2 = OL^2 = OV^2 - K^2.$$

*The square of the distance between the self-corresponding points equals the product of the maximum and minimum segments.*

The maximum and minimum segments are

$$2(K - OV) \text{ and } -2(K + OV).$$

But

$$\begin{aligned} KL^2 &= 4(OV^2 - K^2) \\ &= -4(K + OV)(K - OV) \\ &= EE'\,.\,FF'. \end{aligned}$$

If the self-corresponding points are real, $EE'$ and $FF'$ have the same sign, one is a maximum (numerically) and one a minimum (numerically). If the self-corresponding points are imaginary $EE'$ and $FF'$ have opposite signs, and are then numerically both minima.

*The distance between the vanishing points is equal to the semi-sum of the maximum and minimum segments.*

The maximum and minimum segments are

$$2K - V'V \text{ and } -2K - V'V.$$

Therefore their semi-sum is $-V'V$, the distance between the vanishing points.

*The product of the distances of a pair of corresponding points from their respective vanishing points equals* $-\frac{1}{4^2}$ *the square of the difference of the maximum and minimum segments.*

The difference of the maximum and minimum segments is

$$(2K - V'V) - (2K - V'V) = 4K,$$

$$\therefore\ K^2 = \frac{1}{4^2}\ \text{(difference of segments)}^2.$$

Hence the maximum and minimum segments are such that

(1) They are bisected at the mean point, *i.e.* the mid-point of $VV'$.

(2) Their product equals the square of the distance between the self-conjugate points.

(3) Their semi-sum equals the distance between the vanishing points.

(4) The square of their difference is $(-16)$ times the product of the distances of a pair of corresponding points from their vanishing points.

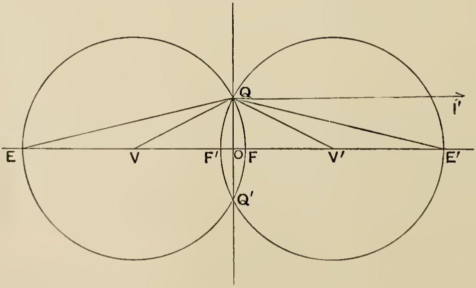
**43.** *If $EE'$ and $FF'$ are the maximum and minimum segments and circles be described on $EF$ and $E'F'$ as diameters, the self-corresponding elements are the limiting points of the system of coaxal circles* determined by these circles, and if the circles intersect at $Q$ and $Q'$, corresponding segments of the ranges subtend equal angles at $Q$ and $Q'$.*

The centres of the circles described on $EF$ and $E'F'$ as diameters are at $V$ and $V'$ the vanishing points and the two circles are equal. If they do not intersect the real points

$$OE \,.\, OF = OV^2 - K^2 = OK^2 = OL^2. \quad \text{(Art. 42.)}$$

Therefore $K$ and $L$ are the common harmonic conjugates of $EF$ and $E'F'$. Hence they are the limiting points of the coaxal system when real. The circle on $LM$ cuts the circles on $EF$ and $E'F'$ orthogonally.

If the circles on $EF$ and $E'F'$ intersect as in the figure in $Q$ and $Q'$ then the angle $E'QE = \pi - F'QF$. Let $I$ be the point at infinity on a parallel to the base through $Q$.



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* See Art. 83.

Then $$I'QE' = QE'E = QEE' = EQV, \qquad \therefore\ I'QV = E'QE.$$

Hence $EE'$, $FF'$, $I'V$ all subtend equal angles or angles whose sum is $\pi$ at $Q$ and also at $Q'$.

If $VP \,.\, V'P' = K^2$ the self-corresponding points are real and the maximum and minimum segments do not exist. Hence:

(1) If the maximum and minimum segments $EE'$ and $FF'$ are imaginary the self-conjugate points are real.

(2) If the maximum and minimum segments are real and the circles on $EF$ and $E'F'$ do not intersect the self-corresponding points are real and are their limiting points.

(3) If the maximum and minimum segments are real and the circles on $EF$ and $E'F'$ intersect in $Q$ and $Q'$ then the self-corresponding points are imaginary, and corresponding points subtend equal angles at $Q$ and $Q'$.

It therefore follows that in the case of two superposed projective ranges there are either

(1) A pair of self-corresponding elements $S$ and $S'$, in which case, if $A$ and $A'$ be a pair of corresponding points, the point $P'$ corresponding to any point $P$ of either range may be found from the fact that

$$(SS'AP) = (SS'A'P') \quad \text{(Example 5, page 76.)}$$

or

(2) There are two points $Q$ and $Q'$ at which pairs of corresponding elements subtend equal angles.

In case (2) it is obvious that the points $Q$ and $Q'$ must be symmetrically situated in regard to the base, and that the maximum and minimum segments are obtained when the arms of the rotated angle are equally inclined to the line $QQ'$.

## Summary of the Properties of two Superposed Projective Ranges.

In the case of two projective ranges on the same base in which $A, B, C \ldots$ correspond to $A', B', C' \ldots$

(*a*) There is a vanishing point $W$ or $V'$ of each range such that

$$WA \,.\, V'A' = WB \,.\, V'B' = \text{constant}.$$

(*b*) There is a point $O$ half-way between the vanishing points which may be called the mean point.

(*c*) There are two self-corresponding points $S$ and $S'$ which may be real or imaginary.

(*d*) If $S$ and $S'$ are imaginary—and sometimes when they are real—there are maximum and minimum segments $KK'$, $LL'$.

(*e*) If $S$ and $S'$ are imaginary, circles described on $KL$, $K'L'$ intersect in two points $Q$ and $Q'$ at which pairs of corresponding points subtend equal angles.

(*f*) If $S$ and $S'$ are real, any pair of corresponding points may be constructed by means of the fact that $(SS'AP)=(SS'A'P')$, where $P$ and $P'$ correspond. If $S$ and $S'$ are imaginary, corresponding points may be obtained by the rotation of a constant angle round $Q$ or $Q'$.

(*g*) The limiting points of the circles described on $KL$ and $K'L'$ as diameters are the self-corresponding points of the ranges.

### 44. Connexion between Ranges determined in different ways.

The different relationships between ranges of points (or pencils of rays) may be defined as follows, viz.

If two ranges (or pencils) are such that one may be deduced from the other by a series of projections, the ranges (or pencils) are said to be *projective.*

If two ranges (or pencils) are such that the anharmonic ratio of any four elements of the one is equal to the anharmonic ratio of the four corresponding elements of the other the ranges (or pencils) are said to be *equianharmonic.*

If two ranges (or pencils) are such that pairs of corresponding elements are determined by means of an algebraic relation, and to any element of the one corresponds one and only one element of the other, then the ranges (or pencils) are said to be *uniquely determined* and to have one to one correspondence.

It has already been shown (Arts. 11 and 35) that the terms projective and equianharmonic, as defined above, when applied to ranges or pencils, are equivalent terms. It will now be shown that the terms equianharmonic and uniquely determined are, in this case, also equivalent. Equianharmonic ranges are (Art. 12) uniquely determined and thus it is only necessary to prove that uniquely determined ranges are equianharmonic.

Let $x$ be the distance of any point of the first range from some origin on its base and $x'$ the distance of the corresponding point of the other from an origin on its base. Then if the points are uniquely determined each from the other, by an algebraic relation, the relation between $x$ and $x'$ must be of the form

$$Axx'+Bx+Cx'+D=0.$$

Therefore $$x=-\frac{Cx'+D}{Ax'+B} \quad \text{..............................(i).}$$

Suppose that $x_1'$, $x_2'$, $x_3'$, $x_4'$ are the coordinates of four points of the second range corresponding to the points of the first range whose coordinates are $x_1, x_2, x_3, x_4$. The anharmonic ratio of these points is

$$\frac{x_1 - x_3}{x_2 - x_3} : \frac{x_1 - x_4}{x_2 - x_4} \quad \text{..........................(ii).}$$

But from (i) $x_1 - x_3 = (x_1' - x_3') \dfrac{AD - CB}{(Ax_1' + B)(Ax_3' + B)}$.

Substituting in (ii) this and the corresponding values for

$$x_2 - x_3, \ x_1 - x_4, \ x_2 - x_4$$

it is seen that

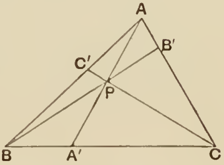
$$\frac{x_1 - x_3}{x_2 - x_3} : \frac{x_1 - x_4}{x_2 - x_4} = \frac{x_1' - x_3'}{x_2' - x_3'} : \frac{x_1' - x_4'}{x_2' - x_4'}$$

which proves the theorem.

The corresponding theorems for pencils may be deduced without difficulty by considering the ranges formed by the pencils on any transversal.

### Determination of Position by Ratio.

If $B$ and $C$ are two given points on a line the position of any point $A'$ on the line is known if the ratio $\dfrac{BA'}{CA'}$ be given. (See Art. 7.)



In a triangle $ABC$ the position of points situated on the three sides may be given by their ratios with respect to the vertices. Ceva's and Menelaus' Theorems may then be stated as follows:

*When three points are situated on the sides of a triangle, if the product of their ratios with respect to the vertices is* $-1$, *the lines joining the points to the vertices are concurrent, and, if the product of their ratios is* $+1$, *the points are collinear.*

## EXAMPLES.

(1) Two sets of four collinear points are such that the points of one set are at distances $2, \frac{1}{2}, -1, +1$ from a fixed point on the line, and those of the other set at distances $-3, \frac{3}{2}, -4, -\frac{39}{7}$ from the point. Prove that two projective four-point ranges can be formed out of the two sets respectively.

(2) $ABCD$ and $A'B'C'D'$ are two ranges of collinear points such that

$$\frac{\lambda}{AD} + \frac{\mu}{A'D'} = \frac{\lambda}{BD} + \frac{\mu}{B'D'} = \frac{\lambda}{CD} + \frac{\mu}{C'D'}.$$

Prove that they have the same anharmonic ratios.

From the given relations $\lambda \left\{\dfrac{1}{AD} - \dfrac{1}{BD}\right\} = -\mu \left\{\dfrac{1}{A'D'} - \dfrac{1}{B'D'}\right\}$,

$$\therefore \frac{1}{AD}\frac{BA}{BD} = -\frac{\mu}{\lambda}\frac{1}{A'D'}\frac{B'A'}{B'D'},$$

so
$$\frac{1}{AD}\frac{CA}{CD} = -\frac{\mu}{\lambda}\frac{1}{A'D'}\frac{C'A'}{C'D'},$$
$$\therefore\ \frac{BA}{BD} : \frac{CA}{CD} = \frac{B'A'}{B'D'} : \frac{C'A'}{C'D'},$$
$$\therefore\ (ADBC) = (A'D'B'C').$$

(3) If $P$ and $P'$ correspond in the two homographic ranges $ABC\ldots$ and $A'B'C'\ldots$ show that
$$\frac{AB \,.\, CP}{A'B'} + \frac{AC \,.\, PB}{A'C'} + \frac{AP \,.\, BC}{A'P'} = 0.$$

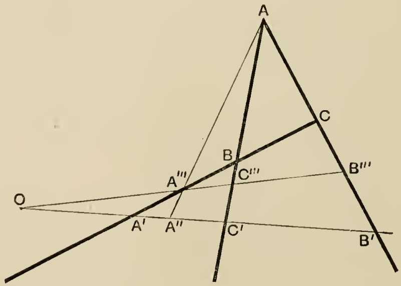
Since the ranges are projective, $(ACBP) = (AC'B'P')$,
$$\therefore\ \frac{AB \,.\, CP}{CB \,.\, AP} = \frac{A'B' \,.\, C'P'}{C'B' \,.\, A'P'}$$
$$= \frac{\dfrac{1}{A'C'} - \dfrac{1}{A'P'}}{\dfrac{1}{A'C'} - \dfrac{1}{A'B'}},$$
$$\therefore\ \frac{1}{A'C'}\{AB \,.\, CP - CB \,.\, AP\} - \frac{1}{A'B'}AB \,.\, CP + \frac{1}{A'P'}CB \,.\, AP = 0,$$
$$\therefore\ \frac{AB \,.\, CP}{A'B'} + \frac{AC \,.\, PB}{A'C'} + \frac{AP \,.\, BC}{A'P'} = 0.$$

(4) If $\Omega$, $\Omega'$ be the double points of two projective ranges and $AA'$ and $BB'$ pairs of corresponding points then $(\Omega A \Omega' A') = (\Omega B \Omega' B')$.

(5) If a straight line meet the sides of $BA$ and $BC$ of a triangle $ABC$ in $C_1$ and $A_1$ respectively, and the lines joining $P$, any point on the line, to $C$ and $A$ meet the opposite sides in $C'$ and $A'$, prove that
$$\frac{AC_1}{BC_1} : \frac{AC'}{BC'} + \frac{CA_1}{BA_1} : \frac{CA'}{BA'} = 1.$$

By projection from $P$, $(ABC_1C') = (A'BA_1C) = (CA_1BA') = 1 - (CBA_1A')$,
$$\therefore\ (ABC_1C') + (CBA_1A') = 1.$$

(6) Draw a line through a given point to meet the sides of a given triangle in three points which with the given point form a range of given anharmonic ratio.



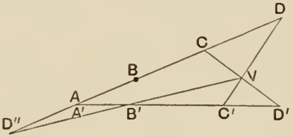
Let $O$ be the given point. Draw any transversal to meet the sides in $A'C'B'$. Take a point $A''$ on this line such that $(OA''C'B')$ has the given anharmonic ratio. Join $A''$ to $A$ to meet $AB$ in $A'''$. Let $OA'''$ meet $AB$ and $AC$ in $C'''$ and $B'''$. Then $OA'''B'''C'''$ is the required transversal.

(7) Prove the following test for collinearity of the three points $O$, $M$, $N$; $OABC$ and $OA'B'C'$ are any two straight lines through $O$; also $AMB'$, $BMA'$, $B'NC$, $BNC'$ are straight lines. Then the required condition for collinearity is that $AA'$, $BB'$, $CC'$ are concurrent.

In Art. 35 $s''$ will not pass through $O$ unless the ranges are in perspective, in which case $AA'$, $BB'$, $CC'$ are concurrent.

(8) Two ranges $ABCD$ and $A'B'C'D'$ on different lines have a common point $A$; $CD'$ and $C'D$ meet at $V$. Show that

$$(AB, CD) \times (AB', C'D') = V(B'B, CD).$$



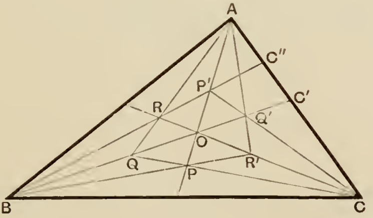
In the figure

$$(AB'C'D') = (V.AB'C'D') = (AD''DC).$$

Therefore

$$(ABCD)(AB'C'D') = (ABCD)(AD''DC)$$
$$= (DCBA)(DCAD'') = (DCBD'') = (BD''DC)$$
$$= (V.B'BCD).$$

(9) $AO$, $BO$, $CO$ connect the vertices of a triangle to $O$; $P$ is taken on $AO$, $PC$ meets $BO$ in $Q$, $QA$ meets $CO$ in $R$, $RB$ meets $AO$ in $P'$, $P'C$ meets $BO$ in $Q'$, and $Q'A$ meets $CO$ in $R'$. Prove that $R'BP$ are collinear.



In the figure

$$(A.BPR'C) = (A.BOQ'C) = (C.BRP'C'') = (A.BQOC')$$
$$= (C.BQOC') = (C.BPR'C').$$

The first and last of these pencils are in perspective and therefore $B$, $P$, $R'$ are collinear.

(10) If $ab'c$, $a'bc'$ be such that the three sides $a$, $c$, $b$ are concurrent and likewise the sides $b'$, $a'$, $c'$ show that the lines joining the opposite vertices, viz., $ac'.a'c$; $ab'.a'b$; $cb'.bc'$ are concurrent.

## Examples on Ceva's and Menelaus' Theorems.

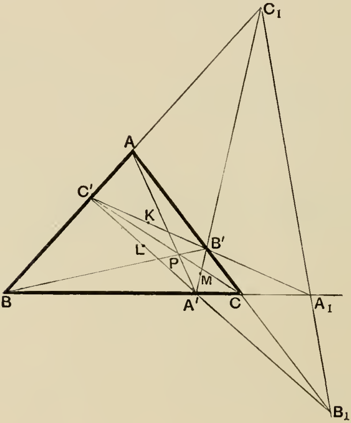
(11) If the lines joining the vertices of a triangle $ABC$ to any point $P$ meet the opposite sides respectively in $A'B'C'$ and $K$, $L$, $M$ are the middle points of $C'B'$, $C'A'$, $A'B'$ respectively, and these lines meet the opposite sides in $A_1$, $B_1$, $C_1$ respectively, prove that

(1) $A_1$, $B_1$, $C_1$ are collinear,

and (2) $AK$, $BL$, $CM$ are concurrent.

(1) Since $A'$, $B'$, $C_1$ are collinear

$$\frac{AC_1}{BC_1}=\frac{1}{\frac{BA'}{CA'}\cdot\frac{CB'}{AB'}} \quad \text{(Menelaus' Theorem).}$$



By multiplying this by the two similar expressions, it is seen that

$$\frac{AC_1}{BC_1}\cdot\frac{BA_1}{CA_1}\cdot\frac{CB_1}{AB_1}=\frac{1}{\left(\frac{BA'}{CA'}\cdot\frac{CB'}{AB'}\cdot\frac{AC'}{BC'}\right)^2}=1.$$

Hence $A_1$, $B_1$, $C_1$ are collinear.

(2) Let $AK$, $BL$, $CM$ meet the opposite sides in $K'$, $L'$, $M'$.

Then $$(C'B'KA_1)=(BCK'A_1).$$

$$\therefore \frac{C'K}{B'K}:\frac{C'A_1}{B'A_1}=\frac{BK'}{CK'}:\frac{BA_1}{CA_1},$$

or $$-1:\frac{C'A_1}{B'A_1}=\frac{BK'}{CK'}:\frac{BA_1}{CA_1}.$$

By multiplying together this and the two similar relations, it is seen that

$$(-1) : \Pi \frac{C'A_1}{B'A_1} = \Pi \frac{BK'}{CK'} : \Pi \frac{BA_1}{CA_1},$$

or

$$(-1) : 1 = \Pi \frac{BK'}{CK'} : 1 \text{ since } A_1,\ B_1,\ C_1 \text{ are collinear.}$$

$$\therefore\ \Pi \frac{BK'}{CK'} = -1.$$

Therefore by Ceva's Theorem $AK'$, $BL'$, $CM'$ are collinear.

(12) From the angular points of a triangle $ABC$ lines $AD$, $BE$, $CF$ are drawn cutting the opposite sides in $D$, $E$, $F$. The lines $AD$, $BE$, $CF$ form a triangle $A'B'C'$.

Prove that
$$\frac{AF.CE.BD}{BF.AE.CD} = \frac{A'C.B'A.C'B}{A'B.B'C.C'A}.$$

Consider the triangle $ABC'$. By Menelaus' Theorem

$$\frac{AF}{BF} = \frac{1}{\dfrac{BA'}{C'A'} \cdot \dfrac{C'B'}{AB'}}.$$

By multiplying together this and the two similar relations,

$$\frac{AF}{BF} \cdot \frac{CE}{AE} \cdot \frac{BD}{CD} = \frac{AB'}{BA'} \cdot \frac{BC'}{CB'} \cdot \frac{CA'}{AC'}.$$

(13) The middle points of the sides of a triangle $ABC$ are $A'$, $B'$, $C'$ and the feet of the perpendicular from the vertices are $D$, $E$, $F$. Prove that if

$$(AEB'C) = (AFC'B) = (BDA'C)$$

the triangle must be equilateral.

The given relationships may be written

$$(ACEB') = (ABFC') = (BCDA') = K \text{ (suppose)}$$

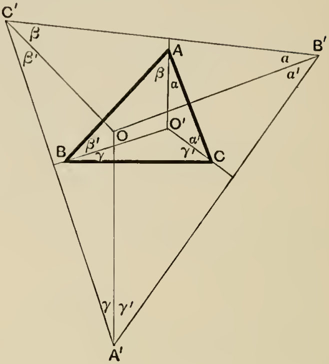
$$\therefore\ \frac{(ABFC')\,(BCDA')}{(ACEB')} = K,$$

$$\therefore\ \frac{\dfrac{AF}{BF} \cdot \dfrac{BD}{CD} \cdot \dfrac{CE}{AE}}{\dfrac{AC'}{BC'} \cdot \dfrac{BA'}{CA'} \cdot \dfrac{CB'}{AB'}} = K, \qquad \therefore\ K = 1.$$

Therefore since $(ACEB') = 1$ $E$ and $B'$ coincide. Similarly, $F$ and $C'$ coincide, and $D$ and $A'$. Hence the triangle is equilateral.

(14) $ABC$, $A'B'C'$, are two triangles such that the perpendiculars from $A$ on $B'C'$, from $B$ on $C'A'$, and from $C$ on $A'B'$ are concurrent. Show that the perpendiculars from $A'$ on $BC$, from $B'$ on $CA$, and from $C'$ on $AB$ are also concurrent.

In the figure, the angles marked $\alpha$, $\alpha'$, $\beta$, $\beta'$, $\gamma$, $\gamma'$ are equal.



By the Sine form of Ceva's Theorem, the condition that the perpendiculars through $ABC$ should be concurrent is

$$\frac{\sin\alpha \sin\beta \sin\gamma}{\sin\alpha' \sin\beta' \sin\gamma'} = 1.$$

This is also the condition that the perpendiculars through $A'B'C'$ should be concurrent.

(15) The triangle $PQR$ is inscribed in the triangle $ABC$ and is in perspective with it: and the triangle $XYZ$ is circumscribed about the triangle $ABC$ and is in perspective with it. Show that the triangles $PQR$, $XYZ$ are in perspective.

Use Sine form of Ceva's Theorem.

(16) $ABC$ is a triangle and $O$ a point within it. Through $ABC$ and within the triangle lines are drawn making with $AB$, $BC$, $CA$ angles equal respectively to $OAC$, $OBA$, $OCB$; prove that these lines meet at a point whose distances from the sides of the triangle are inversely proportional to the corresponding distances of $O$.

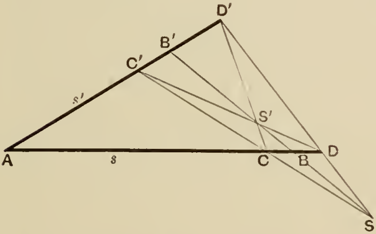
Use Sine form of Ceva's Theorem.

# CHAPTER VII

## HARMONIC FORMS:—HARMONIC PROPERTY OF THE QUADRANGLE AND QUADRILATERAL. HARMONIC PERSPECTIVE.

### 45. Harmonic Ranges and Pencils.

*If two harmonic ranges of four points have a self-corresponding point at the point of intersection of the bases, then the ranges have two centres of perspective.*



Let $ABCD$ and $AB'C'D'$ be the ranges. Let $CD' . C'D$ be $S'$ and $DD' . CC'$ be $S$.

Since

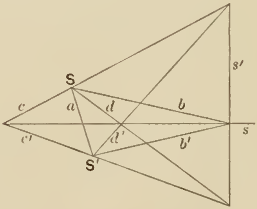
$$(ABCD) = -1 = (AB'C'D')$$

the ranges $ABCD$ and $AB'C'D'$ are projective. Since $A$ is a self-corresponding point they are in perspective.

Therefore $BB'$ passes through $S$.

Similarly $BB'$ passes through $S'$. Hence the theorem is true.

*If two harmonic pencils of four rays have a self-corresponding ray in the line joining their vertices, then the pencils have two axes of perspective.*



Let $abcd$ and $ab'c'd'$ be the pencils. Let $cd' . c'd$ be $s'$ and $dd' . cc'$ be $s$.

Since

$$(abcd) = -1 = (ab'c'd')$$

the pencils $abcd$ and $ab'c'd'$ are projective. Since $a$ is a self-corresponding ray they are in perspective.

Therefore $bb'$ is on $s$.

Similarly $bb'$ is on $s'$. Hence the theorem is true.

Conversely:

*If two ranges of four points, which have a self-corresponding point at the point of intersection of their bases, have also two centres of perspective, then the ranges are harmonic.*

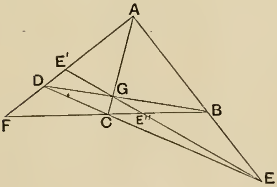
In the figure if $S$ and $S'$ be the centres of perspective,

$$(ABCD) \overset{'}{=} (AB'C'D') = (ABDC).$$

### **46. Harmonic Property of the quadrangle and construction of harmonic conjugates.**

*The distance between any pair of vertices of a quadrangle, which are collinear with a given vertex of its diagonal points triangle, is divided harmonically by that vertex and the opposite side of the diagonal points triangle.*

This is practically the theorem proved in the last article, but on account of its importance the proof is restated.



Let $ABCD$ be the quadrangle, and $E$, $F$, $G$ its diagonal points triangle. Let $GE$ meet $DA$ and $CB$ in $E'$ and $E''$.

Conversely:

*If two pencils of four rays, which have a self-corresponding ray in the line joining their vertices, have also two axes of perspective, then the pencils are harmonic.*

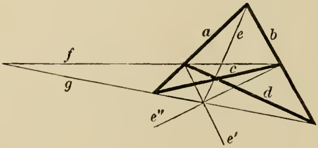
In the figure if $s$ and $s'$ be the axes of perspective,

$$(abcd) = (ab'c'd') = (abdc).$$

### **Harmonic Property of the quadrilateral and construction of harmonic conjugates.**

*The angle between any pair of sides of a quadrilateral, which intersect on a given side of its diagonal triangle, is divided harmonically by that side and the line joining the opposite vertex of the diagonal triangle to the point of intersection of the pair of sides.*

This is practically the theorem proved in the last article, but on account of its importance the proof is restated.



Let $abcd$ be the quadrilateral, and $e$, $f$, $g$ its diagonal triangle. Let the lines joining $ge$ to $da$ and $cb$ be $e'$ and $e''$.

Then projecting from $E$,

$$(FE''CB) = (FE'DA).$$

Projecting from $G$,

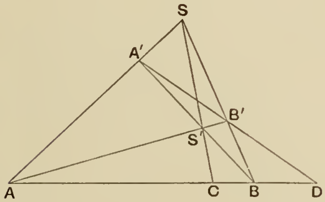
$$(FE''CB) = (FE'AD).$$

Therefore

$$(FE'DA) = (FE'AD).$$

Therefore $A$ and $D$ are harmonic conjugates of $F$ and $E'$.

*Given a range of three points to construct graphically the harmonic conjugate of any one with respect to the other two.*



Let $A, B, C$ be the given points. It is required to construct the harmonic conjugate of $C$ with respect to $A$ and $B$.

Take any two points $S$ and $S'$ collinear with $C$ and join $S$ and $S'$ to $A$ and $B$. Let $AS'.SB$ and $S'B.SA$ be $B'$ and $A'$. Then from the harmonic property of a complete quadrangle the line $A'B'$ meets the base in the required point $D$.

Then taking sections on $e$,

$$(fe''cb) = (fe'da).$$

Taking sections on $g$,

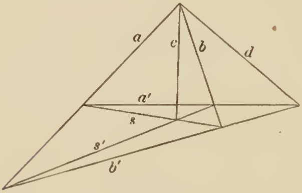
$$(fe''cb) = (fe'ad).$$

Therefore

$$(fe'da) = (fe'ad).$$

Therefore $a$ and $d$ are harmonic conjugates of $f$ and $e'$.

*Given a pencil of three rays to construct graphically the harmonic conjugate of any one with respect to the other two.*



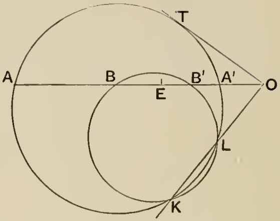
Let $a, b, c$ be the given lines. It is required to construct the harmonic conjugate of $c$ with respect to $a$ and $b$.

Take any two lines $s$ and $s'$ intersecting on $c$ and construct the intersections of $s$ and $s'$ with $a$ and $b$. Let $as'.sb$ and $s'b.sa$ be $b'$ and $a'$. Then from the harmonic property of the complete quadrilateral the line joining $a'b'$ to the vertex is the required ray $d$.

**47.** *To find the common pair of harmonic conjugates of two given pairs of collinear points* $A, A'$ *and* $B, B'$.

Take any point $K$ not on the straight line $ABB'A'$.

Describe circles through $AKA'$ and $BKB'$. These circles will intersect in a second point $L$. Let $KL$ meet $ABB'A'$ in $O$. Draw a tangent $OT$ to one of the circles and measure off distances $OE$ and $OF$ on $AA'$ equal to $OT$.



Then

$$OE^2 = OF^2 = OT^2$$
$$= OA \, . \, OA' = OB \, . \, OB'.$$

Hence $E$ and $F$ are harmonic conjugates of $AA'$ and of $BB'$ and are therefore the required points.

It should be noticed that

(1) $E$ and $F$ are uniquely determined;

(2) $E$ and $F$ are real if $O$ is outside the circles and, for this to be the case, the segments $AA'$ and $BB'$ must not overlap.

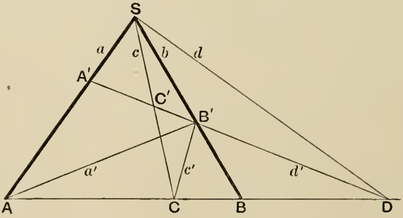
### 48. Conjugate Points—Pole and Polar.

*Conjugate points with respect to two given straight lines* are such that their connector is cut harmonically by the pair of lines.

Thus, in the figure if $(ABCD) = -1$ $C$ and $D$ are conjugates with respect to $a$ and $b$.

*Conjugate lines with respect to two given points* are such that they are harmonic conjugates of the lines joining their point of intersection to the given points.

Thus, in the figure if $(abcd) = -1$ $a$ and $b$ are conjugates with respect to $C$ and $D$.



Hence, it follows that if through any fixed point $D$ a variable transversal be drawn to cut two given lines in $A'$ and $B'$ and a point $C'$ be taken on the line such that $A'B'C'D$ is harmonic ($C'$ being a conjugate of $D$), then the locus of $C'$ is a straight line $c$ passing through the point of intersection of the given lines.

Hence, it follows that if on any fixed line $b$ a variable point $B'$ is taken and lines $c'$ and $d'$ are drawn to the fixed points $C$ and $D$ and a line $a'$ is taken such that the pencil $(a'bc'd')$ is harmonic ($a'$ being the conjugate of $b$), then $a'$ always passes through a fixed point $A$, which is on the line joining

The line $c$ is called the *polar* of $D$ with respect to the given lines.

This line is the locus of the conjugates of $D$.

The point $D$ is called the *pole* of $c$ with respect to the pair of lines.

It should be noticed that

*Every side of the diagonal points triangle of a quadrangle is the polar of the opposite vertex with respect to two pairs of connectors of the four points.*

the given points. The point $A$ is called the *pole* of $b$ with respect to the pair of given points.

This point is the envelope of the conjugates of $b$.

The line $b$ is called the *polar* of $A$ with respect to the pair of points.

*Every vertex of the diagonal triangle of a quadrilateral is the pole of the opposite side with respect to two pairs of intersections of the four lines.*

### 49. Harmonic Perspective.

Any pair of straight lines may be considered to be in Harmonic Perspective with each other, any point being the centre of perspective and the polar of this point with regard to the two lines being the axis.

In the figure of Article 48, let $SA$ and $SB$ be the given lines. Take $D$ as centre of perspective and $c$ the polar of $D$ with regard to $a$ and $b$ as axis of perspective. Then $(DC'B'A')$ is a harmonic range. Therefore $A'$ corresponds to $B'$ and $B'$ to $A'$, in such a harmonic perspective.

Hence, the lines $A'B$ and $AB'$, being corresponding lines, intersect on $SC$ the axis of perspective.

Any quadrilateral (or quadrangle) may be regarded as in Harmonic Perspective with itself, any vertex of the diagonal (or diagonal points) triangle and the opposite side being the centre and axis of perspective.

If the points $ac$ and $bd$ are situated on the side $e$ of the diagonal triangle $efg$, then the poles of $e$ with respect to $a$ and $c$ are on $ac\,.\,fg$, and the poles of $e$ with respect to $d$ and $b$ are on $bd\,.\,fg$. Therefore $fg$ is a common pole of $e$ with respect to $a$ and $c$ and to $d$ and $b$. Hence, with $gf$ as centre and $e$ as axis of harmonic perspective, $a$ and $c$ mutually correspond and as do also $d$ and $b$. Taking different vertices as centres of perspective the lines $a$, $b$, $c$, $d$ will correspond in different pairs.

The following are immediate consequences of the preceding:

*If any straight line $u$ meets the sides $a, b, c, d$ of a quadrilateral in $K, L, M, N$, and $S$ be one of the vertices of its diagonal triangle (viz. the intersection of $ab\,.\,cd$ with $ad\,.\,bc$) and $SK$, $SL$, $SM$, $SN$ meet $c$, $d$, $a$, $b$ in $K'$, $L'$, $M'$, $N'$, then $K'$, $L'$, $M'$, $N'$ are collinear.*

Correlative:

*If any point $U$ be joined to the vertices $A, B, C, D$ of a quadrangle by lines $k, l, m, n$, and $s$ be the third diagonal of the quadrangle, and $sk$, $sl$, $sm$, $sn$ be joined to the opposite vertices of the quadrangle, viz. $C$, $D$, $A$, $B$, by lines $k'$, $l'$, $m'$, $n'$, then $k'$, $l'$, $m'$, $n'$ are concurrent.*

A particular case of the preceding is the following:

*If through the point of intersection of the diagonals of a quadrilateral straight lines are drawn parallel to the four sides to meet the sides, which are respectively opposite to those to which they are drawn parallel, then the four points of intersection are collinear.*

Or correlatively :

*If through the vertices $A$, $B$, $C$, $D$ of a quadrangle any system of parallel lines be drawn to meet the third diagonal in $K$, $L$, $M$, $N$, and $K$, $L$, $M$, $N$ be joined to the opposite vertices of the quadrangle, then these four lines are concurrent.*

### 50. Generalisation of the theorems of Art. 16.

If $AB$, $CD$ are two pairs of harmonic conjugates and $O$ is the middle point of $AB$, then $\infty$, the point at infinity on the base, is the harmonic conjugate of $O$ with respect to $AB$. Hence, if $AB$, $CD$, $O\infty$ are projected into $A'B'$, $C'D'$, $J'I'$, respectively, $A'$, $B'$ are the common harmonic conjugates of $C'D'$ and $J'I'$. Hence the theorem of Art. 16 may be generalised as follows :

$$(b)\quad \frac{CA}{CD}=\frac{CO}{CB}, \qquad \therefore\ \frac{CA}{CD}:\frac{\infty A}{\infty D}=\frac{CO}{CB}:\frac{\infty O}{\infty B},$$

$$\therefore\ (ADC\infty)=(OBC\infty), \qquad \therefore\ (A'D'C'I')=(J'B'C'I'),$$

$$\therefore\ (A'C'D'I')=(J'C'B'I'),$$

$$\therefore\ \frac{A'D'\,.\,B'C'}{A'I'\,.\,B'J'}=\frac{D'C'}{I'J'}$$

$$(c)\quad OC\,.\,OD=OA^2, \qquad \therefore\ \frac{OC}{OA}=\frac{OA}{OD}, \qquad \therefore\ (CAO\infty)=(ADO\infty),$$

$$\therefore\ (C'A'J'I')=(A'D'J'I'),$$

$$\therefore\ \frac{J'C'\,.\,J'D'}{(J'A')^2}=\frac{I'C'\,.\,I'D'}{(I'A')^2}.$$

$$(d)\quad \frac{OC}{OD}=\left(\frac{AC}{AD}\right)^2, \qquad \therefore\ (CDO\infty)=(CDA\infty)^2,$$

$$\therefore\ (C'D'J'I')=(C'D'A'I')^2,$$

$$\therefore\ \frac{C'J'\,.\,C'I'}{D'J'\,.\,D'I'}=\left(\frac{C'A'}{D'A'}\right)^2.$$

Extensions of the above theorems may also be obtained by introducing $O'$ the middle point of $DC$. In this case $O$ and $O'$ are the harmonic conjugates of the point at infinity $\infty$ on the line with respect to $AB$ and $CD$ respectively.

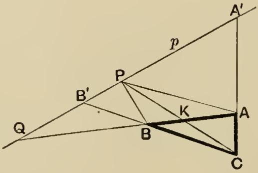
### EXAMPLES.

(1) Given a line $p$ and three fixed points $A$, $B$, $C$, to find a point $P$ on $p$ such that $p$ and $PC$ are harmonic conjugates of $PA$ and $PB$.

Let $AB$ meet $p$ in $Q$. Take $K$ the harmonic conjugate of $Q$ with respect to $AB$. Let $KC$ meet $p$ in $P$.

Then $(P\,.\,QKAB)$ is harmonic and therefore $P$ is the required point.

$P$ may also be obtained as the harmonic conjugate of $Q$ with respect to $A'$, $B'$, the points where $CA$ and $CB$ meet $p$.



(2) Given three points $A$, $B$, $C$ and any other point $P$ to draw through $P$ a line which shall be the harmonic conjugate of $PC$ with respect to $PA$ and $PB$.

Let $PC$ meet $AB$ in $K$. On $AB$ take $Q$ the harmonic conjugate of $K$ with respect to $AB$. Then $PQ$ is the required line.

(3) Given four lines $l_1$, $l_2$, $l_3$, $l_4$, and four points $P_1$, $P_2$, $P_3$, $P_4$ on them such that the connectors of $P_1$, $P_2$, $P_3$, $P_4$ to $C$ are harmonic conjugates of $l_1$, $l_2$, $l_3$, $l_4$ with respect to the connectors of $P_1$, $P_2$, $P_3$, $P_4$ to $A$ and $B$, prove that the pencil $(C \,.\, P_1P_2P_3P_4)$ is projective with the range formed on $AB$ by the lines $l_1$, $l_2$, $l_3$, $l_4$.

(4) The lines joining any point $O$ to four collinear points $A$, $B$, $C$, $D$ cut any transversal through $D$ in $\alpha$, $\beta$, $\gamma$, $D$, respectively, and $B\gamma$ meets $OA$ in $O'$, $O'\beta$ meets $AB$ in $B'$. Prove that if $(AB,\ CD)=-1$ then $AB' \,.\, CB=2 \,.\, AC \,.\, BB'$.

It is necessary to prove that $\dfrac{AB'}{BB'} : \dfrac{AC}{BC} = -2$ or that $(ABB'C)=-2$.

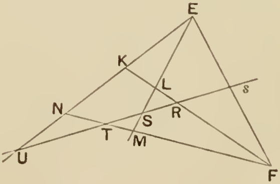
But
$$(ABCD)=(\alpha\beta\gamma D)=(AB'BD)=-1,$$
$$\therefore\ (ABB'D)=2 \text{ and } (ABCD)=-1,$$
$$\therefore\ (ABB'C)=\frac{(ABB'D)}{(ABCD)}=-2.$$

(5) $KLMN$ is a simple quadrangle and $A$, $B$ are two points such that $(K \,.\, LNAB)$, $(L \,.\, KMAB)$, $(M \,.\, LNAB)$ are harmonic.

Prove that:

(i) $(N \,.\, MKAB)$ is harmonic, and

(ii) $A$, $B$ must be situated on the third diagonal of the quadrangle.



If the transversal $AB$ meets the sides as in the figure in $RSTU$,
$$(ABUR)=-1,$$
$$(ABRS)=-1,$$
$$(ABST)=-1.$$
$$\therefore\ \frac{AU}{BU}=-\frac{AR}{BR}=\frac{AS}{BS}=-\frac{AT}{BT}.$$
Therefore the pencil $(N \,.\, KMAB)$ is harmonic. Since $\dfrac{AU}{BU}=\dfrac{AS}{BS}$, $U$ and $S$ coincide or $AB$ passes through $E$. Similarly $AB$ passes through $F$. Hence $A$, $B$ are any pair of harmonic conjugates of $E$ and $F$.

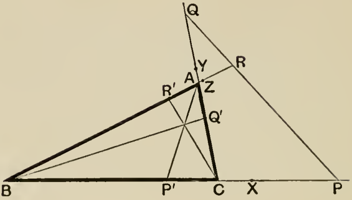
(6) If $ABCD$ be a quadrangle and $AC \,.\, BD$ bisects $DB$, then the connector of $AB \,.\, CD$ to $AD \,.\, CB$ is parallel to $BD$, or $ABCD$ is a parallelogram.

(7) If a transversal cut the sides $BC$, $CA$, $AB$ of a triangle in $P$, $Q$, $R$ respectively, then

(i) if $P'$, $Q'$, $R'$ be the harmonic conjugates of $P$, $Q$, $R$ with respect to $BC$, $CA$, $AB$ respectively, $AP'$, $BQ'$, $CR'$ meet in a point, and

(ii) if $X$, $Y$, $Z$ be the middle points of $PP'$, $QQ'$, $RR'$ respectively, $X$, $Y$, $Z$ lie on a straight line.

Since
$$(BCP'P)=-1,$$
$$\therefore\ \frac{P'B}{P'C}=-\frac{PB}{PC}.$$



Similarly

$$\frac{Q'C}{Q'A} = -\frac{QC}{QA},$$

and

$$\frac{RA}{R'B} = -\frac{RA}{RB}.$$

$$\therefore \frac{P'B.Q'C.R'A}{P'C.Q'A.R'B} = (-1)^3 \frac{PB.QC.RA}{PC.QA.RB}$$

$= -1$ by Menelaus' Theorem.

Therefore by Ceva's Theorem $AP'$, $BQ'$, $CR'$ are concurrent.

Since $\dfrac{XB}{XC} = \left(\dfrac{PB}{PC}\right)^2$ it follows that

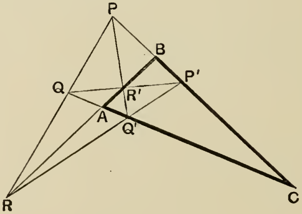
$$\frac{XB.YC.ZA}{XC.YA.ZB} = (-1)^6 \left(\frac{PB.QC.RA}{PC.QA.RB}\right)^2 = +1.$$

Therefore by Menelaus' Theorem $X$, $Y$, $Z$ are collinear.

Given that $AP'$, $BQ'$, $CR'$ are concurrent, the converse theorem can be proved.

The point of concurrency of $AP'$, $BQ'$, $CR'$ is termed *the pole of the line PQR with respect to the triangle.*

This result should be compared with Example 11, Chapter VI.



(8) The middle points of the diagonals of a quadrilateral are collinear.

In the figure the sides of the triangle $ABC$ are cut by a transversal $PQR$ and from the quadrangle $QQ'P'P$, $PP'$, $QQ'$, $RR'$ divide the sides of $ABC$ harmonically. Hence the middle points of $PP'$, $QQ'$, $RR'$ are collinear. But $PP'$, $QQ'$, $RR'$ are the diagonals of the quadrilateral $QPQ'P'$. Hence the theorem is true.

(9) The three straight lines joining the vertices $A$, $B$, $C$ of a triangle to a point cut the opposite sides in points $P$, $Q$, $R$ respectively, and $R'$ is the point on $BA$ which divides it externally in the same ratio as $R$ divides it internally. Prove that the straight line joining the middle points of $AP$, $BQ$ passes through the middle point in $CR'$.

Let $S$ be the point to which the vertices are joined. The line $PQ$ will pass through $R'$. Then $BQ$, $AP$, $R'C$ are the diagonals of the quadrilateral, $BA$, $AQ$, $QP$, $PB$. Hence the middle points are collinear.

(10) Show that the three poles of a straight line with respect to the three pairs of points of intersection of four given straight lines (*i.e.* the ends of diagonals of the quadrilateral) lie upon another straight line conjugate to the first straight line with respect to each of the three pairs of points.

Take the diagonal triangle $ABC$ as triangle of reference and let its sides meet the line in $A'$, $B'$, $C'$. Let the ratios of $R$, $T$, $S$ three collinear ends of diagonals be $x_1, y_1, z_1$, then those of the other ends of the diagonals $R'$, $T'$, $S'$ will be

$$-x_1, \ -y_1, \ -z_1.$$

Let the ratios of $A'$, $B'$, $C'$ be $a'$, $b'$, $c'$; then if $a_2$, $b_2$, $c_2$ be those of the harmonic conjugates of these points with respect to $RR'$, $SS'$, $TT'$,

$$a_2=\frac{x_1^2}{a'};\quad b_2=\frac{y_1^2}{b'};\quad c_2=\frac{z_1^2}{c'},$$

$$\therefore\ a_2b_2c_2=\frac{x_1^2y_1^2z_1^2}{a'b'c'}=\frac{1^2}{1}=1.$$

$\therefore$ these points are collinear.

This theorem is an extension of Example 8.

(11) If through any point $P$ two lines be drawn to meet the sides of a triangle $ABC$ in $C'$, $B'$ and $C''$, $B''$ and points $A'$, $A''$ be taken on $BC$ such that the triangles $A'B'C'$ and $A''B''C''$ are in perspective with $ABC$. Then if $Q$ be the point of intersection of $A'B'$ and $A''B''$, $P$ and $Q$ are collinear with $B$ and the pencil $A\,.\,PQBC$ is harmonic.

Consider the pencils $(B'\,.\,ABC'A')$ and $(B''\,.\,ABC''A'')$ which have a self-corresponding ray $AB'B''$ and whose other rays intersect in $B$, $P$, $Q$. Let $BB'\,.\,CC'$ be $S'$ and $BB''\,.\,CC''$ be $S''$. The rays of $(B'\,.\,ABC'A')$ intersect $AB$ in $A$, $B$, $C'$ and a point $C_1$. But from the quadrangle $B'S'A'C$ this range is harmonic;

$$\therefore\ (B'\,.\,ABC'A')=-1\,;\ \text{so}\ (B''\,.\,ABC''A'')=-1.$$

Therefore $P$, $B$, $Q$ are collinear and $(A\,.\,PQBC)$ is harmonic.

(12) Through a point $O$ in the plane of a triangle $ABC$ three straight lines are drawn meeting the sides $BC$, $CA$, $AB$ in $D_1E_1F_1$; $D_2E_2F_2$; $D_3E_3F_3$, respectively, so that the ranges $(OD_1E_1F_1)$, $(OE_2F_2D_2)$, $(OF_3D_3E_3)$ are all harmonic. Show that $D_1$, $E_2$, $F_3$ are collinear and also $D_1E_3F_2$; $D_2E_1F_3$; $D_3E_2F_1$.

(13) If through one vertex $C$ of a triangle $ABC$ a chord be drawn to meet the opposite side in $K$ and two points $C'$ and $K'$ be taken on $CK$ such that $C$, $C'$ are harmonic conjugates of $KK'$, then the pencils $AC'$ and $AK'$ are projective.

For $(CC'KK')=-1$, $\therefore\ (CKC'K')=2$, $\therefore\ \dfrac{CC'}{KC'}=2\,\dfrac{CK'}{KK'}$.

Hence the ranges $K'$ and $C'$ are projective.

(14) If a transversal meets the sides $BC$, $CA$, $AB$ of a triangle, $ABC$, in points $K$, $L$ and $M$, and points $K'$, $L'$, $M'$ are taken on the sides such that

$$(BCKK')=(CALL')=(ABMM'),$$

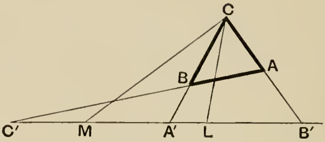
prove that the poles of $KLM$ with respect to the triangle $ABC$ and all triangles $K'L'M'$ coincide.

This theorem may be proved directly or the straight line $KLM$ may be projected into the line at infinity and the triangle into an equilateral triangle in which case the theorem becomes the following, which can be easily proved to be true.

If on the sides of an equilateral triangle $ABC$ points $K'$, $L'$, $M'$ are taken such that they divide the sides in the same ratios, then the centroids of the triangles $ABC$ and $K'L'M'$ coincide.

(15) If $ABC$ be a triangle and $L$ and $M$ are two points such that $(C\,.\,LMAB)$ is harmonic, then $(A\,.\,LMCB) = -(B\,.\,LMCA)$.

Let $AB$, $BC$, $CA$ meet $LM$ in $C'$, $A'$, and $B'$. Take $L$, $M$ as points of reference and let $a$, $b$, $c$ be the ratios of $A'$, $B'$, $C'$.



Then

$$(C\,.\,LMAB) = (LMB'A') = \frac{b}{a},$$

$$\therefore\ b = -a,$$

$$(A\,.\,LMCB) = (LMB'C') = \frac{b}{c},$$

$$(B\,.\,LMCA) = (LMA'C') = \frac{a}{c} = -\frac{b}{c},$$

$$\therefore\ (A\,.\,LMCB) = -(B\,.\,LMCA).$$

(16) If $ABC$ be a triangle and $L$ and $M$ are a pair of points such that

$(C\,.\,LMAB) = \lambda$, then $(A\,.\,LMCB) = \lambda\,(B\,.\,LMCA)$.

# CHAPTER VIII

## INVOLUTION

**51.** There are two ways of defining an involution:

**First definition of an involution.**

*If pairs of points* $AA'$, $BB'$, $CC'$, ... *are taken on a straight line, termed the base, such that if* $O$ *be a given point on the base*

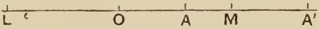
$$OA \,.\, OA' = OB \,.\, OB' = OC \,.\, OC' = a \textit{ constant quantity},$$

*then the points* $AA'$, $BB'$, $CC'$, ... *are said to form an involution of which* $A$ *and* $A'$, $B$ *and* $B'$, $C$ *and* $C'$, ... *are termed conjugate points.*

The properties of an involution differ according as the constant is positive or negative. When the constant is positive the involution may be said to be of the *first kind* and when negative of the *second kind.*

Involution of the first kind.

If $OA \,.\, OA'$ is a positive quantity, $A$ and $A'$ are situated on the same side of $O$, and $OA$ and $OA'$ must be both positive or both negative. If a length $OK$ be taken such that $OA \,.\, OA' = OK^2$, two points $L$ and $M$ are obtained at equal distances $OK$ from $O$ in the positive and in the negative direction respectively such that each is its own conjugate. These points are termed the *double points* of the involution. Since $O$ is the middle point of $LM$ and $OM^2 = OL^2 = OA \,.\, OA'$, it follows that $L$ and $M$ are harmonic conjugates of $A$ and $A'$. Similarly $L$ and $M$ are harmonic conjugates of $B$ and $B'$ and of every other pair of conjugate points. Hence pairs of conjugate points of the involution may in this case be constructed as pairs of harmonic conjugates of $L$ and $M$ the double points of the involution.



Involution of the second kind.

If $OA \,.\, OA'$ is a negative quantity, $A$ and $A'$ are situated on different sides of $O$, and $OA$ and $OA'$ must have different signs. If double points are sought for, as in the previous case, the equation $OA \cdot OA' = -OK^2$ is obtained. That is to say, $L$ and $M$ are at distances $+\sqrt{-1} \cdot OK$ and $-\sqrt{-1} \cdot OK$ from $O$. Hence $L$ and $M$, the double points, are what may be termed imaginary points. Assuming such points to exist on the base they are equally distant from $O$, and consequently, since $OL^2 = OM^2 = OA \cdot OA'$, $A$ and $A'$ may be looked upon as harmonic conjugates of these points. There are, in this case, a pair of conjugate points equally distant on either side from $O$.

In both kinds of involution the point $O$ corresponds to the point at infinity on the base. This point is called the centre of the involution.

*The anharmonic ratio of any four points of an involution is equal to that of their four conjugates.*

To prove this it must be shown that

$$(ABCD) = (A'B'C'D').$$

But $OA = K \dfrac{1}{OA'}$; $OC = K \dfrac{1}{OC'}$, where $K$ is a constant,

$$\therefore \quad AC = -K \frac{A'C'}{OA' \cdot OC'}.$$

Similar relations hold for the distances between other pairs of points.

$$\therefore \quad \frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} \cdot \frac{OB'}{OA'} : \frac{A'D'}{B'D'} \cdot \frac{OB'}{OA'}$$

$$= \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'},$$

$$\therefore \quad (ABCD) = (A'B'C'D').$$

**Second definition of an involution.**

*If two superposed projective ranges have a pair of elements, which mutually correspond to each other, then every other pair of elements mutually correspond and the ranges are said to form an involution.*

If $A$ and $A'$ are the pair of mutually corresponding elements and $B'$ of the second corresponds to $B$ of the first range, then $P$ the point of the first which corresponds to $B$ of the second is given by

$$(AA'BP) = (A'AB'B).$$

But

$$(A'AB'B) = (AA'BB'),$$

$$\therefore \quad (AA'BP) = (AA'BB'),$$

therefore $P$ coincides with $B'$. This proves the theorem involved in the definition.

This result may also be obtained for Arts. 10 and 37. For if $V$ and $V'$ be the vanishing points of the ranges and $P$ and $P'$ mutually correspond, $VP \,.\, V'P' = \text{constant} = VP' \,.\, V'P$. Therefore $V$ and $V'$ coincide in a point $O$. Hence the equation $VP \,.\, V'P' = \text{constant}$ becomes $OP \,.\, OP' = \text{constant}$ and each pair of corresponding points mutually correspond.

The relation on which the first definition of an involution depends may be deduced at once as follows:

If $O$ be the point corresponding to the point at infinity ($\infty$) on the involution, then by definition

$$(AB'\infty\, O) = (A'BO\infty),$$

$$\therefore \quad \frac{A\infty}{B'\infty} : \frac{AO}{B'O} = \frac{A'O}{BO} : \frac{A'\infty}{B\infty},$$

therefore $$AO \,.\, A'O = BO \,.\, B'O = \text{a constant}.$$

If the constant be positive, two points $L$ and $M$ may be taken on either side of $O$ such that $OL^2 = OM^2 = OA \,.\, OA'$ and $L$ and $M$ are then harmonic conjugates of $AA'$ and of every other pair of conjugate points.

### 52. Involution Pencil.

Since an involution is a particular case of two superposed projective ranges, an involution is projected from any centre upon any line—in the plane of the centre of projection and the base—into another involution. The pencil formed by joining the points of an involution to any point is termed an involution pencil and is cut by every line in its plane in another involution. It should be noticed that the centre of an involution is not projected into the centre of the corresponding involution, but the double points are projected into double points. The new centre, when the double points are real, may be determined as their mid-point. The centre is always the conjugate of the point at infinity on the base.

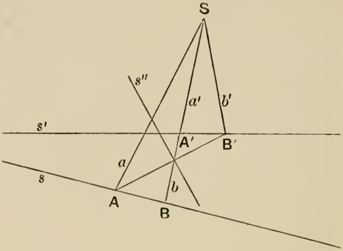
On reference to Art. 48 it will be seen that a system of collinear conjugate points with regard to two straight lines form an involution.

The case in which two projective pencils become an involution may be graphically considered by reference to Art. 35.

Generally in two projective pencils with the same vertex $S$ any ray $a'$ or $b$ corresponds to a different ray $a$ or $b'$ according as it is considered to belong to the one or the other of the pencils. Let the pencil $ab\ldots$ determine a range $AB\ldots$ on $s$ and the pencil $a'b'\ldots$ a range $A'B'\ldots$ on $s'$, where $b$ and $a'$ are the same line.

Then $AB' \,.\, BA'$ will intersect on some fixed line $s''$. (Art. 35.)

Now if $s''$ does not pass through $S$ the lines $SA$ and $SB'$ cannot coincide. If $s''$ passes through $S$, $AB'$ must pass through $S$ because it passes through the point where $s''$ meets $A'B$. In this case the lines $SB'$ and $SA$ do coincide. Hence $SA'B$ has the same corresponding ray $SB'A$ to whichever pencil it is supposed to belong. Similarly, every ray has the same corresponding ray to whichever pencil it is supposed to belong. In this case, the two projective pencils become an involution pencil.



**53.** An involution is completely determined by

(1) Two pairs of conjugate points,
(2) The two double points,
(3) The centre and a double point,
(4) A pair of conjugate points and the centre,
(5) A pair of conjugate points and a double point.

If two pairs of conjugate points $AA'$, $BB'$ be given, the conjugate of any point $C$ is uniquely determined by the relation

$$(AA'BC')=(A'AB'C).$$

A graphic construction for $C'$ is given (Art. 59).

From the fact that $(ABA'C')=(A'B'AC)$,

$$\frac{AA'}{BA'}:\frac{AC'}{BC'}=\frac{A'A}{B'A}:\frac{A'C}{B'C},$$

$$\therefore\quad AB'.BC'.CA'+A'B.B'C.C'A=0.$$

This is a relation which must be satisfied by three pairs of conjugate points, $AA'$, $BB'$, $CC'$ of an involution.

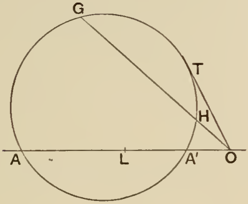
Also since $$(AA'BC')=(A'AB'C),$$

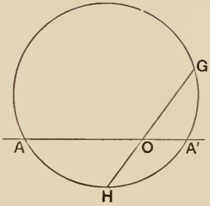
$$\therefore\quad \frac{AB}{A'B}:\frac{AC'}{A'C'}=\frac{A'B'}{AB'}:\frac{A'C}{AC}.$$

Hence $$\frac{AB}{A'B}\cdot\frac{AB'}{A'B'}=\frac{AC'}{A'C'}\;\frac{AC}{A'C}=\text{constant}.$$

*Hence if pairs of conjugate points of an involution are determined by ratios $bb'$, $cc'$, ..., $A$ and $A'$, any pair of conjugate points, being the points of reference, $bb'=cc'=a$ constant.*

*To construct on a given base an involution having given the centre and the constant of the involution.*



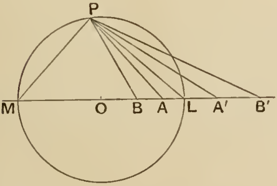


Take any point $G$ not on the base, and on $OG$, where $O$ is the centre, take a point $H$ such that $OG \,.\, OH$ equals the constant of the involution, $H$ and $G$ being on the same or on opposite sides of $O$ according as the constant is positive or negative. If $A$ be any point on the base, the circle through $AGH$ will meet the base in $A'$ the point conjugate to $A$, for $OA \,.\, OA' = OH \,.\, OG =$ the constant of the involution.

When the constant of the involution is positive a tangent $OT$ can be drawn to the circle and the double points $L$ and $M$ are at distances $OT$ from $O$.

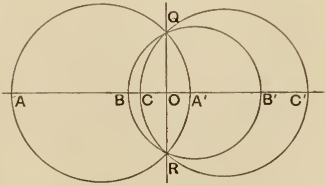
When the constant of the involution is negative, $OG$ and $OH$ can be taken equally distant from $O$ on the perpendicular through $O$ to the base and in this case every pair of conjugate points $A$ and $A'$ subtend a right angle at $G$ and $H$.

The following illustrates the difference between an involution with real double points and one with imaginary double points.



Let the double points $M$ and $L$ be real.

Describe a circle on $ML$ as diameter.



Let the double points be imaginary.

Take $Q$ and $R$ on the perpendicular through $O$ such that $OQ = -OR$ and $OQ \,.\, OR = OA \,.\, OA'$.

Join $P$, any point on this circle, to $ML$, and to pairs of conjugate points $AA'$, $BB'$, etc.

Since $PM$ and $PL$ are at right angles and the pencils $(P . MLAA')$ and $(P . MLBB')$ are harmonic, $PL$ bisects the angles $APA'$, $BPB'$, etc. (Art. 16.)

Therefore the angles $APB$ and $A'PB'$ are equal.

Therefore corresponding segments of an involution with real double points subtend equal angles at any point of the circle described on the line joining the double points as diameter.

Describe circles through $QR$. They will have their centres on the base and determine on the base pairs of conjugate points $AA'$, $BB'$, etc.

Since $AQA'$ and $BQB'$ are right angles, therefore the angles $AQB$ and $A'QB'$ are equal.

Therefore corresponding segments of an involution with imaginary double points subtend equal angles at two points, and pairs of corresponding points of the involution subtend right angles at these points.

When the double points are real, if $P$ be any point on the circle described on the line joining them as diameter,

$$\frac{AP}{A'P} = \frac{AL}{A'L} = \frac{AM}{A'M}.$$

**54.** It has been shown that an involution may be looked upon as a particular case of two superposed projective ranges. On reference to Art. 43 it will be seen that in the case of an involution the points $W$, $V'$ and $O$ coincide and that the properties there proved for two superposed projective ranges become as follows:

(*a*) $OA . OA' = OB . OB' =$ constant.

(*b*) The self-corresponding points $S$ and $S'$ may be real or imaginary.

(*c*) If $S$ and $S'$ are real, corresponding points are harmonic conjugates of $S$ and $S'$.

If $S$ and $S'$ are imaginary, corresponding points may be constructed by the rotation of a constant angle, which in this case is a right angle, round $Q$ or $Q'$ (Art. 53), two points situated on the perpendicular through $O$.

(*d*) If $S$ and $S'$ are imaginary, there are two coincident minimum segments, $KK'$, and a circle described on $KK'$ as diameter passes through $Q$ and $Q'$.

There are two special forms of an involution:

(1) If one of the double points is at infinity, the other double point bisects the distance between every pair of conjugate points.

This follows from the fact that every pair of conjugate points are harmonic conjugates of the double points.

(2) If the two double points coincide, every point on the line is a conjugate of the point at which the double points coincide. This is the particular case of a harmonic range dealt with in Art. 15.

These special cases are of importance in connexion with the Conic Section.

**55.** *To find the common pair of conjugate elements of two superposed involutions.*

Let $O$ and $O'$ be the centres of the two involutions.

(1) Let the double points $EF$ and $E'F'$ of both involutions be real.

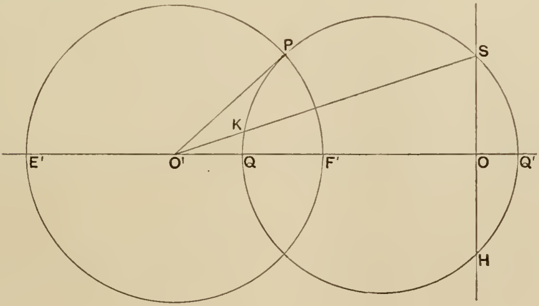
Then the common pair of conjugates of the two involutions are the common harmonic conjugates of $EF$ and $E'F'$. (Art. 51.) These are real unless $EF$ and $E'F'$ overlap. (Art. 47.)

(2) Let the double points of both involutions be imaginary.

Construct points $S$, $H$ equally distant vertically above and below $O$ such that any circle through $S$ and $H$ determines conjugate points of the involution centre $O$. Similarly construct $S'$, $H'$ for the other involution. Then a circle can be described through $SHS'H'$ and this circle determines the common pair of conjugates of the involutions.

(3) Let the double points of the involution centre $O$ be imaginary and those of the involution centre $O'$ be real.

Construct $S$ and $H$ as above for the involution centre $O$ and let $E'$ and $F'$ be the double points of the involution centre $O'$.



Join $O'S$ and on it take a point $K$ such that $O'F'^2 = O'K \,.\, O'S$. Describe a circle through $K$, $S$, and $H$ to meet $OO'$ in $Q$, $Q'$. $Q$ and $Q'$

are the required points, for they are conjugates of the involution centre $O$, since the circle which determines them passes through $S$ and $H$ and they are conjugates of the involution centre $O'$ since

$$O'Q \,.\, O'Q' = O'K \,.\, O'S = O'F'^2.$$

The only case in which the common conjugates can be imaginary is when both involutions have real double elements, which overlap.

The common conjugates of two involution pencils can be obtained from the common conjugates of the involutions which they determine on any line.

The construction of the double points of an involution is given in Chapter XVII, Art. 110.

*Given two pairs of lines $a$, $a'$ and $b$, $b'$ to find the pair of common harmonic conjugates of $a$, $a'$ and $b$, $b'$ which are parallel.*

Through $aa'$ draw a pair of rays $b_1$, $b_1'$ parallel to $b$, $b'$. Construct $e$ and $f$ the common harmonic conjugates of $aa'$ and $b_1b_1'$ (Art. 47). The rays through $aa'$ and $bb'$ parallel to $e$ and $f$ are the parallel common harmonic conjugates of $a$, $a'$ and $b$, $b'$.

## 56. Involution property of the quadrangle and quadrilateral.

*The three pairs of lines, which can be drawn through four points, are cut by any transversal in three pairs of conjugate points of an involution.*

Let $S$, $T$, $Q$, $R$ be the four given points and $AA'$, $BB'$, $CC'$ the pairs of points in which their connectors are met by any transversal $s$. Let $SQ \,.\, TR$ be the point $P$.

Since the ranges obtained by projecting $RPTA$ from $S$ and from $Q$ on to $s$ are projective,

$$(C'A'BA) = (B'A'CA) = (CAB'A').$$

Therefore $AA'$, $BB'$, $CC'$ are pairs of conjugate points of an involution.

If the transversal be drawn as

*The three pairs of points, in which four straight lines intersect, determine by their connectors with any point three pairs of conjugate rays of an involution pencil.*

Let $s$, $t$, $q$, $r$ be the four given lines and $aa'$, $bb'$, $cc'$ the pairs of lines joining their intersections to any point $S$. Let $sq \,.\, tr$ be the line $p$.

Since the pencils obtained by joining $S$ to the points of intersection of $s$ and $q$ with $rpta$ are projective,

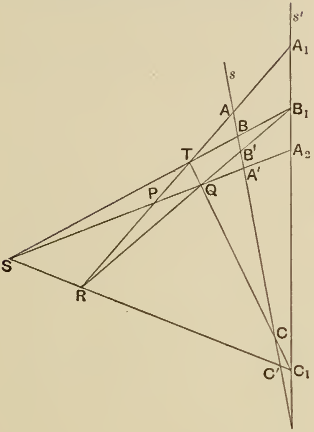
$$(c'a'ba) = (b'a'ca) = (cab'a').$$

Therefore $aa'$, $bb'$, $cc'$ are pairs of conjugate rays of an involution pencil.

If the point $S$ be situated at $S'$,

$s'$ to pass through $TS.QR$ and $TQ.RS$ as in the figure,

$$(C_1A_2B_1A_1) = (C_1A_1B_1A_2).$$



Hence $A_1$ and $A_2$ are harmonic conjugates of $B_1$ and $C_1$.

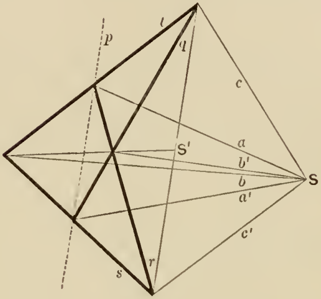
*Cor.*:

If $AA'$, $BB'$ and $C$ are given, $C'$ is fixed, since the points $AA'$, $BB'$, $CC'$ form an involution.

Hence, *if five of the sides of a quadrangle pass through five fixed collinear points, the sixth side must pass through a fixed point collinear with the other five points.* (See Art. 13 ($b$).)

the point of intersection of $ts.qr$ and $tq.rs$, $b'$ coincides with $b$ and $c'$ with $c$.

$$\therefore\ (ca'ba) = (caba').$$



Hence $a$ and $a'$ are harmonic conjugates of $c$ and $b$.

*Cor.*:

If $aa'$, $bb'$ and $c$ are given, $c'$ is fixed, since the lines $aa'$, $bb'$, $cc'$ form an involution.

Hence, *if five of the vertices of a quadrilateral lie on five fixed concurrent straight lines, the sixth vertex must lie on a fixed line concurrent with the other five lines.* (See Art. 13 ($b$).)

*Graphic construction of conjugate elements of an involution.*

If two pairs of conjugate points $A$ and $A'$ and $B$ and $B'$ are given, the theorem of the last article enables us to construct $C'$ the conjugate of any element $C$.

If two pairs of conjugate rays $a$ and $a'$ and $b$ and $b'$ are given, the theorem of the last article enables us to construct $c'$ the conjugate of any element $c$.

Through $C$ draw an arbitrary line on which take any two points $Q$ and $T$.

Join $AT$, $B'Q$ to intersect in $R$, and $BT$, $A'Q$ to intersect in $S$.

Then $SR$ meets the base in the required point $C'$.

On $c$ take any arbitrary point through which draw any two lines $q$ and $t$.

Join $at$ and $b'q$ by a line $r$, and $bt$ and $a'q$ by a line $s$.

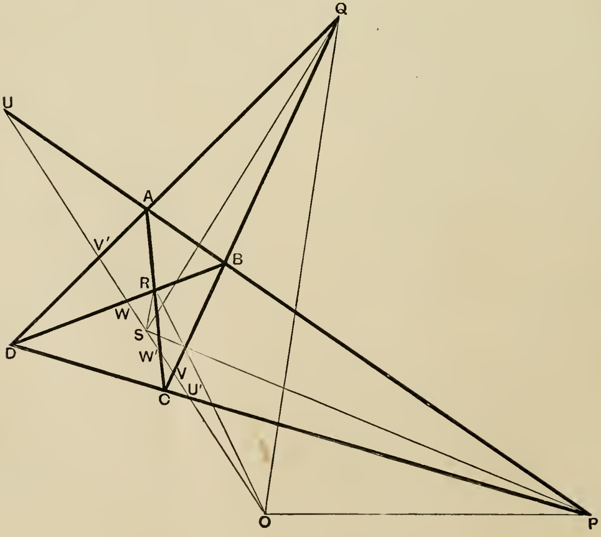
Then the line joining $rs$ to the vertex of the pencil is the required ray $c'$.

**57.** *Involutions determined by pairs of lines.*

In Art. 48 conjugate points with respect to pairs of lines were defined. Collinear pairs of conjugate points form an involution and a pair of straight lines determines an involution on every line in their plane. This involution is formed by pairs of harmonic conjugates of the two points of intersection of the line with the given pair of lines.

There are two lines on which any two given pairs of lines determine the same involution, namely, the third pair of connectors of their four points of intersection.

*If $ABCD$ be a quadrangle, $P$, $Q$, $R$ its diagonal points and $O$ any point in its plane, then the harmonic conjugates of $OP$, $OQ$, $OR$, with respect to the sides of the quadrangle which meet at $P$, $Q$, $R$, respectively, are concurrent.*



Let the harmonic conjugates of $OP$ and $OQ$ with regard to the sides which meet at $P$ and $Q$ intersect in $S$.

Let $OS$ meet the pairs of opposite sides in $UU'$, $VV'$, $WW'$. These points form an involution.

But $(OSUU')$ and $(OSVV')$ are harmonic, and therefore $O$ and $S$ are the double points of this involution. Therefore $(OSWW')$ is harmonic. Therefore $RS$ is the harmonic conjugate of $RO$ with respect to $RA$, $RB$.

$O$ and $S'$ are conjugate points with respect to the three pairs of opposite sides of the quadrangle.

The correlative theorem is as follows:

*If abcd be a quadrilateral, p, q, r its diagonals and o any line in its plane, then the harmonic conjugates of op, oq, or, with respect to the vertices of the quadrilateral which lie on p, q, r respectively, are collinear.*

This may be proved by the correlative method. Example 10, Chapter VII, is a restatement of the theorem and Example 8, Chapter VII, is a particular case.

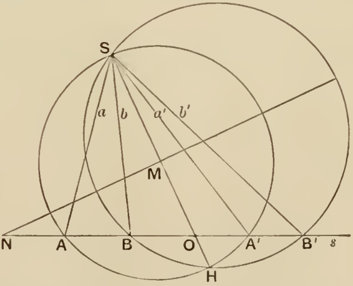
These theorems are particular cases of an important theorem proved in Art. 117.

**58. In every involution pencil one pair of conjugate rays is at right angles, and if more than one pair of conjugate rays are at right angles every pair of conjugate rays is at right angles.**

Let $aa'$, $bb'$ be the pairs of conjugate rays, which determine the involution. Let any transversal $s$ meet the rays in $A$, $A'$, $B$, $B'$. Let $O$ be the centre of this involution. Describe a circle through $ASA'$ to meet $SO$ in $H$. Then since

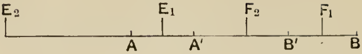
$$OS \,.\, OH = OA \,.\, OA' = OB \,.\, OB',$$

the circle through $SBB'$ will also pass through $H$. Hence conjugate points of the involution are determined by circles through $S$ and $H$.



If $a$ and $a'$ and also $b$ and $b'$ are at right angles, the centres of the circles $SAA'$ and $SBB'$ will lie on $s$ and the centre of every circle through $S$ and $H$ will also be on this line. In this case every pair of conjugate rays will be at right angles.

If $aa'$ and $bb'$ are not at right angles, take $M$ the middle point of $SH$ and erect a perpendicular through $M$ to meet $s$ at $N$. Describe a circle with centre $N$ through $S$ and $H$ to meet $s$ in $C$ and $C'$. Then $s$ is a diameter of this circle and $C$ and $C'$ subtend a right angle at $S$.



**59.** Four collinear points $A$, $A'$, $B$, $B'$ determine three involutions: viz.

(1) that in which $AA'$, $BB'$ are pairs of conjugate points.

(2) that in which $AB$, $A'B'$ are pairs of conjugate points.

(3) „ „ „ $AB'$, $BA'$ „ „ „ „ „

In two cases the involutions will not be overlapping, viz. in the figure in cases (1) and (2). The double points of these involutions are real.

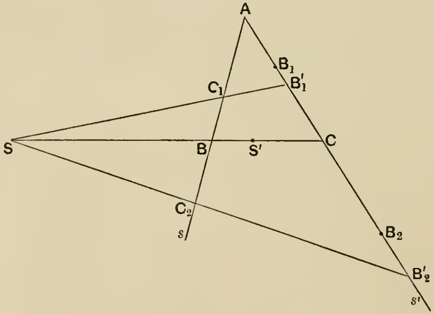
In one case the involution will be overlapping, viz. in the figure in case (3). The double points of this involution are imaginary.

$E_1$ and $F_1$ the common harmonic conjugates of $AA'$ and $BB'$, *i.e.* the double points of (1), are a pair of real conjugate points of (2) and of (3). (Example 1, page 105.)

$E_2$ and $F_2$ the common harmonic conjugates of $AB$, $A'B'$, *i.e.* the double points of (2), are a pair of real conjugate points of (1) and of (3).

$E_3$ and $F_3$ the common harmonic conjugates of $AB'$, $BA'$, *i.e.* the imaginary double points of (3), are a pair of imaginary conjugate points of (2) and of (1).

**60. Every pair of involutions in the same plane are in perspective: if both the involutions have real or both imaginary double points, the perspective is real: if one has real and the other imaginary double points, the perspective is imaginary.**



Let the involutions be situated on two lines $s$ and $s'$ which intersect in $A$ and let $B$ and $C$ be the points of the two involutions which correspond to $A$. Let $C_1C_2$ be any pair of conjugate points of the involution on $AB$ and $B_1B_2$ any pair of conjugate points of the involution on $AC$. Then, if $c_1$, $c_2$, $b_1$, $b_2$ be the ratios respectively of $C_1$, $C_2$, $B_1$, $B_2$ referred to $ABC$, $c_1 . c_2 = K$ and $b_1 . b_2 = K'$, where $K$ and $K'$ are constants for different pairs of conjugate points of the involutions. (Art. 53.)

On $BC$ take two points $S$ and $S'$ such that their ratios are

$$+\frac{1}{\sqrt{KK'}} \text{ and } -\frac{1}{\sqrt{KK'}}.$$

These points are harmonic conjugates of $B$ and $C$. Join $S$ to $C_1$ and $C_2$ to meet $AC$ in $B_1'$ and $B_2'$. Let the ratios of these points be $b_1'$ and $b_2'$.

Then, by Menelaus' Theorem,

$$\frac{1}{\sqrt{KK'}} c_1 b_1' = 1 \quad \text{and} \quad \frac{1}{\sqrt{KK'}} c_2 b_2' = 1.$$

Therefore
$$\frac{c_1 c_2 b_1' b_2'}{KK'} = 1 \quad \text{and} \quad b_1' b_2' = \frac{KK'}{c_1 c_2} = K'.$$

Hence $B_1'$ and $B_2'$ are a pair of conjugate points of the involution on $AC$.

Similarly, if $S'$ be joined to $C_1$ and $C_2$ to meet $AC$ in $B_1''$ and $B_2''$, these are a pair of conjugate points of the involution on $AC$.

Hence if any pair of conjugate points of either involution be joined to $S$ or $S'$ their connectors will meet the other base in a pair of conjugate points of the involution on that base, and the two involutions are in perspective with either $S$ or $S'$ for centre of perspective.

It should be noticed that as long as the involutions are both of the same kind the points $S$ and $S'$ are real.

If $E$ and $F$ are the double points of the involution on $AB$ and $K$ and $L$ the double points of the involution on $AC$, the ratios of these points are $\sqrt{K}$, $-\sqrt{K}$, $\sqrt{K'}$, $-\sqrt{K'}$.

Therefore, since
$$\frac{1}{\sqrt{KK'}} \sqrt{K} \sqrt{K'} = 1, \quad S, E \text{ and } K \text{ are collinear,}$$
$$\frac{1}{\sqrt{KK'}} (-\sqrt{K})(-\sqrt{K'}) = 1, \quad S, F \text{ and } L \quad ,, \qquad ,,$$
$$-\frac{1}{\sqrt{KK'}} (\sqrt{K})(-\sqrt{K'}) = 1, \quad S', E \text{ and } L \quad ,, \qquad ,,$$
$$-\frac{1}{\sqrt{KK'}} (-\sqrt{K})(\sqrt{K'}) = 1, \quad S', F \text{ and } K \quad ,, \qquad ,,$$

Hence if the double points of the involutions are real the double points of one are projected into the double points of the other from either $S$ or $S'$.

Since
$$\frac{1}{\sqrt{KK'}} (\sqrt{K})(-\sqrt{K'}) = -1, \quad \therefore AS, CE, BL \text{ are concurrent,}$$
$$\frac{1}{\sqrt{KK'}} (-\sqrt{K})(\sqrt{K'}) = -1, \quad \therefore AS, CF, BK \quad ,, \qquad ,,$$
$$-\frac{1}{\sqrt{KK'}} (\sqrt{K})(\sqrt{K'}) = -1, \quad \therefore AS', CE, BK \quad ,, \qquad ,,$$
$$-\frac{1}{\sqrt{KK'}} (-\sqrt{K})(-\sqrt{K'}) = -1, \quad \therefore AS', CF, BL \quad ,, \qquad ,,$$

The correlative theorem is as follows:

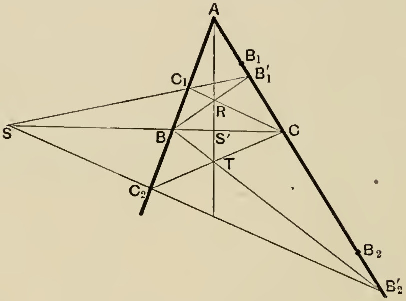
*Every pair of involution pencils in the same plane are in perspective: if both the pencils have real or both imaginary double rays, the perspective is real: if one has real and the other imaginary double rays, the perspective is imaginary.*

Let the centres of the involutions be $B$ and $C$ and let $BA$ and $CA$ be the rays corresponding to $BC$ in the two involutions. Let a pair of rays of the involution, centre $C$, meet $AB$ in $C_1$ and $C_2$ and a pair in the involution centre $B$ meet $AC$ in $B_1$ and $B_2$. With the notation of the correlative theorem draw two lines $AS$ and $AS'$ such that the ratios of $S$ and $S'$, the points where they meet $BC$, are $+\frac{1}{\sqrt{KK'}}$ and $-\frac{1}{\sqrt{KK'}}$. Let $CC_1$ and $CC_2$ meet $AS'$ in $R$ and $T$. Join $BR$ and $BT$ to meet $AC$ in $B_1'$ and $B_2'$. Since $AS'$, $CC_1$, $BB_1'$ are concurrent,

$$-\frac{1}{\sqrt{KK'}}\cdot c_1b_1' = -1,$$

and since $AT$, $CC_2$, $BB_2'$ are concurrent, $-\frac{1}{\sqrt{KK'}}\cdot c_2b_2' = -1$,

$$\therefore \frac{c_1b_1'c_2b_2'}{KK'}=1, \quad \therefore b_1'b_2'=K'.$$



Therefore the lines $BB_1'$ and $BB_2'$ are conjugate rays of the involution whose centre is $B$. Hence the involutions are in perspective with $AS'$ as axis. Similarly they are in perspective with $AS$ as axis.

### 61. Analytical formulae connected with an involution.

If the notation of Art. 14 be used and $B$ and $C$ are taken as the points of reference and $x_1x_2$ ; $x_3x_4$ ; $x_5x_6$, the ratios of $A_1A_2$ ; $A_3A_4$ ; $A_5A_6$, are given by

$$a'x^2 + 2h'x + b' = 0,$$
$$a''x^2 + 2h''x + b'' = 0,$$
$$a'''x^2 + 2h'''x + b''' = 0,$$

the following results are obtained :

(1) The condition that $BC$ ; $A_1A_2$ ; $A_3A_4$ should be three pairs of conjugate points of an involution is

$$x_1x_2 = x_3x_4.$$

(2) The condition that $A_1A_2$ ; $A_3A_4$ ; $A_5A_6$ should be three pairs of conjugate points of an involution is

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1+x_2 & x_3+x_4 & x_5+x_6 \\ x_1x_2 & x_3x_4 & x_5x_6 \end{vmatrix} = 0.$$

(3) The condition (1) that $BC$; $A_1A_2$; $A_3A_4$ should be three pairs of conjugate points of an involution may also be written

$$b'a''-b''a'=0.$$

(4) The condition (2) that $A_1A_2$; $A_3A_4$; $A_5A_6$ should be three pairs of conjugate points of an involution may also be written

$$\begin{vmatrix} a' & h' & b' \\ a'' & h'' & b'' \\ a''' & h''' & b''' \end{vmatrix}=0.$$

(5) The condition that $A_1$, $A_2$ $(x_1, x_2)$ should be a pair of conjugate points of the involution determined by $B$, $A_3$ $(0, x_3)$ and $C$, $A_4$ $(\infty, x_4)$ is

$$x_1x_2-(x_1+x_2)x_4+x_3x_4=0.$$

(6) In each case the double points of the involution are given by making $x_1=x_2=x$, where $x$ stands for the ratio of either of the required double points.

(7) If $A_c(a)$, $A_b(a')$ be the conjugates of $C$ and $B$ in an involution, the double points of the involution are given by

$$x^2-2xa'+aa'=0.$$

## EXAMPLES.

### Involution in connexion with its base.

(1) If $C$ and $C'$ are the common harmonic conjugates of $AB$ and $A'B'$, then $AA'$, $BB'$, $CC'$ are conjugate points of an involution.

In this case $\qquad (ABCC')=-1=(A'B'C'C)$,

$$\therefore\ (ABCC')=(A'B'C'C),$$

$\therefore$ $AA'$, $BB'$, $CC'$ form an involution.

(2) If $A$ and $A'$ be any fixed pair of conjugate points of an involution and $P$ and $P'$ any other pair of conjugate points of the involution, then the common harmonic conjugates of $AP$ and $A'P'$ and of $AP'$ and $A'P$ form, for different pairs of conjugate points $P$ and $P'$, an involution of which $A$ and $A'$ are a pair of conjugate points.

(3) $XY$, $AB$, $CD$ are an involution on a straight line and $(AB, CD)=-1$. If $(PX, AB)=-1$, prove that $(PY, CD)=-1$.

(4) If $A$, $B$, $C$, $X$, $Y$, $P$ are six points on a straight line such that

$$(ABCP)=(ABCX).(ABCY),$$

prove that $AB$, $XY$, $CP$ are three pairs of conjugate points of an involution.

(5) If $LL'$, $NN'$ be two pairs of conjugate points of an involution of which $P$ and $Q$ are the double points, then

$$\frac{LP}{NP}\cdot\frac{LQ}{NQ}=\frac{LN'}{NN'}\cdot\frac{LL'}{NL'},$$

and $\qquad (LNPN')=(LNLQ')$.

(6) $PP'$, $QQ'$, $RR'$ are points on a straight line such that $PP'$, $QR$, $Q'R'$ and also $PP'$, $QR'$, $Q'R$ are pairs of conjugate points of an involution. Prove that $P$, $P'$ are the double points of the involution of which $QQ'$, $RR'$ are pairs of conjugate points.

In this case $(PP'QQ')=(P'PRR')$ and $(PP'QQ')=(P'PR'R)$,

$$\therefore\ (P'PRR')=(P'PR'R)=-1=(PP'QQ').$$

(7) If $E$ and $G$ are any two fixed points and $A$ and $A'$ are any pair of conjugate points of a given involution, then the harmonic conjugates of $E$ and of $G$ with respect to $AA'$ describe two projective ranges.

Let $E'$ and $G'$ be the harmonic conjugates of $E$ and $G$ with respect to $AA'$ and $M$ and $N$ the double points of the involution. Then

$$(AA'MN)=-1\,;\quad (AA'EE')=-1\,;\quad (AA'GG')=-1.$$

Therefore $MN$, $EE'$, $GG'$ are pairs of conjugate points of an involution.

Therefore $$(NMEG)=(MNE'G'),$$

and the ranges described by $E'$ and $G'$ are projective. This proof assumes $N$ and $M$ to be real.

(8) The harmonic conjugates of a variable point with regard to four given pairs of conjugate points of an involution have a constant anharmonic ratio.

(9) $Q$ and $Q'$ are the harmonic conjugates of any point $P$ with respect to $AA'$ and $BB'$ respectively. Prove

(*a*) if $AA'BB'$ is harmonic, that $AA'$, $BB'$, $QQ'$ form an involution.

(*b*) if $AA'$, $BB'$, $QQ'$ form an involution, then $AA'BB'$ is harmonic; and if $R$ and $R'$ be the harmonic conjugates of $Q'$ and $Q$ with respect to $AA'$ and $BB'$ respectively, then $R$ and $R'$ coincide.

(*c*) if another point $P'$ determines $Q$ and $Q'$ as its harmonic conjugates with respect to $BB'$ and $AA'$ respectively, then $AA'$, $BB'$, $PP'$ form an involution.

If $Q$ and $Q'$ be taken as origin, so that the ratio of $Q$ is 0 and that of $Q'$ $\infty$, let the ratios of $P$, $P'$, $A$, $A'$, $B$, $B'$ be $p$, $p'$, $a$, $a'$, $b$, $b'$ respectively.

Since $QPAA'$ is harmonic, $p(a+a')=2aa'$ ........................................(1).

Since $Q'PBB'$ is harmonic, $(b+b')=2p$ ........................................(2).

(*a*) If $AA'BB'$ is harmonic,

$$(a+a')(b+b')=2aa'+2bb'=4a'a \text{ from (1) and (2).}$$

$$\therefore\ aa'=bb'.$$

$\therefore$ $AA'$, $BB'$, $QQ'$ form an involution.

(*b*) If $AA'$, $BB'$, $QQ'$ form an involution, $aa'=bb'$.

$\therefore$ from (1) and (2), $(a+a')(b+b')=4aa'=2aa'+2bb'$.

Let $r$ and $r'$ be the ratios of $R$ and $R'$. Then since $(QR'BB')$ and $(Q'RAA')$ are harmonic $r'(b+b')=2bb'$ and $(a+a')=2r$.

$$\therefore\ r'(a+a')(b+b')=4rbb'.$$

$$\therefore\ r'aa'=rbb',\quad \therefore\ r=r'.$$

(*c*) In this case $$(a+a')=2p'.$$

Therefore $$pp'=\frac{(a+a')(b+b')}{4}=aa'.$$

Hence $P$ and $P'$ are a pair of conjugate points of the involution in part (*a*).

(10) If $AA'$, $BB'$, $CC'$ are three pairs of conjugate points of an involution and $AA'$, $BB'$ form a harmonic range, then the harmonic conjugate of $C$ with respect to $AA'$ and the harmonic conjugate of $C'$ with respect to $BB'$ coincide.

(11) If $X_1Y_1$, $X_2Y_2$, $X_3Y_3$ be three segments dividing $AB$ harmonically and $C_1$, $C_2$, $C_3$ be their middle points, and $P$ be any collinear point, then

$$(PX_1^2+PY_1^2)\,C_2C_3+(PX_2^2+PY_2^2)\,C_3C_1+(PX_3^2+PY_3^2)\,C_1C_2$$

is constant for all positions of $P$.

The given expression equals

$$\begin{vmatrix} 1 & 1 & 1 \\ PC_1 & PC_2 & PC_3 \\ PX_1^2+PY_1^2 & PX_2^2+PY_2^2 & PX_3^2+PY_3^2 \end{vmatrix}$$

$$=\tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ PX_1+PY_1 & PX_2+PY_2 & PX_3+PY_3 \\ (PX_1+PY_1)^2 & (PX_2+PY_2)^2 & (PX_3+PY_3)^2 \end{vmatrix}$$

$$-\begin{vmatrix} 1 & 1 & 1 \\ PX_1+PY & PX_2+PY_2 & PX_3+PY_3 \\ PX_1\,.\,PY_1 & PX_2\,.\,PY_2 & PX_3\,.\,PY_3 \end{vmatrix}$$

The second of these determinants is zero, since relation (2) Art. 61 holds when $x_1$, $x_2$, ... are the distances of three pairs of conjugate points of an involution from any origin $P$.

Hence the expression $=\frac{1}{2}\,\Pi\,(PX_1+PY_1-PX_2-PY_2)$

$=\frac{1}{2}\,\Pi\,(X_2X_1+Y_2Y_1)$, which is independent of $P$.

(12) If $X_1X_2$ and $Y_1Y_2$ be any two pairs of conjugate points of an involution and $\dfrac{\{(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)\}^2}{(X_1X_2Y_1Y_2)}$ is constant, prove that for different positions of $A_1$, $B_1$ describes one or other of two ranges projective with that described by $A_1$.

(13) If three pairs of conjugate rays of an involution pencil contain equal angles, prove that all pairs of conjugate rays of the pencil are at right angles.

### Involution in connexion with the quadrangle and triangle.

(14) The perpendiculars drawn from any point to the pairs of opposite sides of a complete quadrangle form a pencil in involution.

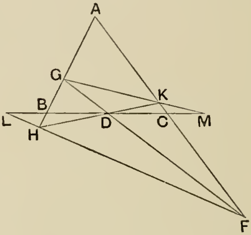
Consider the points in which the sides of the quadrangle meet the line at infinity. These points form an involution. Hence the pencil formed by drawing through any point lines parallel to the sides of the quadrangle is an involution pencil. Hence the pencil formed by the perpendiculars from any point on the sides is also an involution pencil, the angles between corresponding rays of the two pencils being equal.

(15) If three parallel lines are drawn through the diagonal points of a quadrangle, show that their harmonic conjugates, with respect to the pairs of sides of the quadrangle, which meet at these diagonal points, are concurrent. See Art. 57.

(16) If $AA'$, $BB'$, $CC'$ be the pairs of opposite vertices of a quadrilateral and one pair of opposite sides $AB$ and $A'B'$ subtend equal angles at a point $P$, at which the vertices subtend an involution with real double rays, then every other pair of opposite sides subtend equal angles at $P$.

Join $A$, $B$, $C$, $A'$, $B'$, $C'$ to $S$ by $a$, $b$, $c$, $a'$, $b'$, $c'$. Then $aa'$, $bb'$, $cc'$ are pairs of conjugate rays of an involution. Since $APB = A'PB'$ the double rays are the common bisectors of the angles $APA'$ and $BPB'$. Since these are at right angles they must be the bisectors of the angle $CPC'$. Hence the theorem is true.

(17) Prove that if $ABC$ be a triangle and $D$ the middle point of the base $BC$ and $HDK$ be a straight line cutting $AB$ in $H$ and $AC$ in $K$, and $FDG$ be a straight line cutting $AB$ in $G$ and $AC$ in $F$ and $HF$ and $GK$ cut $BC$ in $L$ and $M$, then $D$ is the middle point of $LM$.



Consider the quadrangle $GKFH$. $LM$, $BC$ are conjugate points of an involution of which $D$ is a double point. $D$ bisects the distance between $BC$. Therefore

$$(BCD\infty) = -1.$$

Therefore $$(LMD\infty) = -1.$$

Therefore $D$ bisects $LM$.

(18) Three pairs of lines are drawn through the vertices $A$, $B$, $C$ of a triangle to meet in two points $P$ and $Q$; show that any line cut in involution by the three pairs of lines passes through a fixed point.

(19) If the sides $BC$, $CA$, $AB$ of a triangle are cut respectively by a transversal in $P'$, $Q'$, $R'$, and if $P$, $Q$, $R$, which are points lying on the same transversal, form with $P'$, $Q'$, $R'$ three conjugate pairs of an involution, prove that $AP$, $BQ$, $CR$ are concurrent.

This is the converse of the involution property of a quadrangle.

(20) Perpendiculars are drawn from the vertices of a triangle and from three collinear points on the sides to a straight line lying in the plane of the triangle. Prove that the perpendiculars meet the straight line in a system of points in involution.

(21) Through any point $O$ in the plane of a triangle $ABC$ are drawn $OA'$, $OB'$ $OC'$ bisecting the supplements of the angles $BOC$, $COA$, $AOB$, and meeting $BC$, $CA$, $AB$ in $A'$, $B'$, $C'$ respectively. Show that $A'$, $B'$, $C'$ are collinear and that the six lines $OA$, $OB$, $OC$, $OA'$, $OB'$, $OC'$ form a pencil in involution.

Since $\frac{AC'}{BC'} = \frac{OA}{OB}$ the first part follows from Menelaus' Theorem and the second from the involution property of the quadrilateral.

(22) $A$, $B$, $C$, $D$, $E$ are five coplanar points. Construct a line parallel to $AB$ which shall be cut by $AC$ and $BC$, $AD$ and $BD$, $AE$ and $BE$ in pairs of points of an involution.

(23) Find the double points of an involution in the case when the transversal, cutting a quadrangle, passes through the intersection of a pair of opposite sides of the quadrangle.

### Involution in connexion with the circle.

(24) Prove that if circles be drawn through pairs of points of an involution and through a common external point they will meet in a second fixed point.

(25) Show that if four circles cut a line in pairs of corresponding points of an involution they have a common orthogonal circle.

Let the circles meet the base in $AA'$, $BB'$, $CC'$, $DD'$. Then if $O$ be the centre of the involution, $OA \,.\, OA' = OB \,.\, OB' = K^2$. Hence a circle with centre $O$ and radius $K$ will cut the circles orthogonally. If the double points of the involution are imaginary, the circle will be imaginary.

(26) Find in an overlapping involution the pair of conjugates which are harmonic conjugates of a given pair of conjugates.

Let $AA'$ be the pair of conjugates whose harmonic conjugates are required and $BB'$ any other pair of conjugates. Describe circles on $AA'$ and $BB'$ as diameters to intersect at $O$. Then $AOA'$ is a right angle. The bisectors of the angle $AOA'$ will meet the base in the required points.

(27) If four circles pass through a given point $O$, the connectors of this point $O$ to their six other points of intersection form an involution pencil.

The lines through the six points of intersection of the circles, perpendicular to the lines connecting these points to $O$, pass three by three through four points $A$, $B$, $C$, $D$ one on each of the circles. These six lines perpendicular to the connectors to $O$ therefore form a quadrangle, and the lines through $O$ are therefore perpendicular to the sides of a quadrangle and form an involution pencil.

(28) If four circles touch a given line their six common tangents meet the given line in three pairs of conjugate points of an involution.

This is the correlative of Example 27.

(29) The three circles which can be described on the three diagonals of a quadrilateral as diameters intersect in a pair of points.

Let $AA'$, $BB'$, $CC'$ be the ends of the diagonals. Describe circles on $AA'$, $BB'$ as diameters. Let them intersect in $P$ and $Q$. Then $PA$, $PA'$ and $PB$, $PB'$ are mutually at right angles. Hence, the involution formed by joining $P$ to $AA'$, $BB'$, $CC'$ has therefore two pairs of conjugate rays at right angles. Therefore the third pair $PC$ and $PC'$ are at right angles. Therefore the circle described on $CC'$ as diameter will pass through $P$. Similarly it passes through $Q$.

Since the centres of three such circles are collinear, the middle points of the diagonal are collinear.

(30) The orthocentres of the four triangles, which can be formed by the four sides of a quadrilateral, lie on the line joining the points of intersection of the three circles described on its diagonals as diameters.

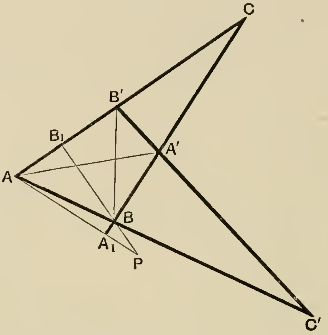
Let $AA'$, $BB'$, $CC'$ be the diagonals of the quadrilateral.

Let the circle on $AA'$ meet $CB$ in $A_1$ and the circle on $BB'$ meet $CA$ in $B_1$.

Then $AA_1C$ and $AB_1B$ are right angles and $AA_1$ and $BB_1$ meet in $P$ the orthocentre of $ABC$.

Since $AA_1BB_1$ are concyclic, $PA \, . \, PA_1 = PB \, . \, PB_1$.

Therefore the tangents from $P$ to the circles on $AA'$ and $BB'$ are equal. Hence as will be proved, Art. 83, $P$ is on the line joining the points of intersection of the circles.



# CHAPTER IX

## COPLANAR FIGURES. PROBLEMS OF THE FIRST DEGREE

**62.** In any plane two figures may be either

(1) identical,
(2) in plane perspective,
(3) projective,
or (4) not projective.

In case (2) every pair of corresponding points are collinear with some point $S$ and every pair of corresponding lines intersect on some line $s$.

It is not sufficient that all pairs of corresponding points should be collinear with some point $S$ for, if it were, any point might be taken as $S$ and lines might be drawn through $S$ to meet the lines in the two figures. Any pairs of points upon these lines might then be taken as pairs of corresponding points, but although all pairs of corresponding points would thus be collinear with $S$, the figures would not be in perspective with $S$ as centre of perspective, for in this case the points of intersection of corresponding lines would not be corresponding points. Also the lines joining corresponding points would not intersect on a given line $s$. In this connexion, Art. 63 should however be noticed.

In case (3) the term projective implies that the figures can be deduced from each other by a series of projections. It was shown in Art. 21 that the characteristics of two such figures are:

(*a*) To every point corresponds a point and to collinear points correspond collinear points.

(*b*) The anharmonic ratio of four collinear points is the same as the anharmonic ratio of the corresponding points.

(*c*) To every line corresponds a line and to concurrent lines correspond concurrent lines.

(*d*) The anharmonic ratio of four concurrent lines is the same as the anharmonic ratio of the corresponding lines.

(*e*) The line joining two points corresponds to the line joining the corresponding points. The point of intersection of two lines corresponds to the point of intersection of the corresponding lines.

In this chapter the difference between the above cases will be considered and also the displacements in one plane of a figure by which it may be brought into one of the above relationships with another figure.

**63.** *Given two figures of points, viz.* $A$, $B$, $C$, ... *and* $A'$, $B'$, $C'$, ... *situated in the same plane, in which* $AA'$, $BB'$, $CC'$, ... *are regarded as corresponding points, and the lines joining pairs of corresponding points as corresponding lines, then if all pairs of corresponding lines intersect in collinear points, the two figures are in plane perspective.*

Consider any three pairs of corresponding points $A$, $B$, $C$ and $A'$, $B'$, $C'$. Since

$$BC . B'C' ; \ CA . C'A' ; \ AB . A'B'$$

are collinear, therefore $AA'$, $BB'$, $CC'$ are concurrent. (Art. 13 (*a*).)

Similarly, if $D$ and $D'$ be any other pair of corresponding points, $AA'$, $BB'$, $DD'$ are concurrent. Therefore not only do all pairs of corresponding lines intersect on a given line but all pairs of corresponding points are collinear with a given point, viz. $AA' . BB'$. Hence the figures are in plane perspective.

*Given two figures of lines, viz.* $a$, $b$, $c$, ... *and* $a'$, $b'$, $c'$, ... *situated in the same plane, in which* $aa'$, $bb'$, $cc'$, ... *are regarded as corresponding lines, and the points of intersection of pairs of corresponding lines as corresponding points, then if all pairs of corresponding points are collinear with the same point, the two figures are in plane perspective.*

Consider any three pairs of corresponding lines $a$, $b$, $c$ and $a'$, $b'$, $c'$. Since

$$bc . b'c' ; \ ca . c'a' ; \ ab . a'b'$$

are concurrent, therefore $aa'$, $bb'$, $cc'$ are collinear. (Art. 13 (*a*).)

Similarly, if $d$ and $d'$ be any other pair of corresponding lines, $aa'$, $bb'$, $dd'$ are collinear. Therefore not only are all pairs of corresponding points collinear with a given point but all pairs of corresponding lines intersect on a given line, viz. $aa' . bb'$. Hence the figures are in plane perspective.

Hence, *if in two projective figures corresponding lines intersect in collinear points or corresponding lines are concurrent at a fixed point, the figures are in plane perspective.*

**64.** *Given any two lines $k$ and $l$ intersecting in $E$, any two lines $k'$ and $l'$ intersecting in $E'$, $S$ any point on $EE'$, and $AA'$, $BB'$, $CC'$, $DD'$ pairs of points collinear with $S$, $A$ and $B$ on $k$, $C$ and $D$ on $l$, $A'$ and $B'$ on $k'$, $C'$ and $D'$ on $l'$; then two figures can be constructed in plane perspective in which the points $A$, $B$, $C$, $D$, $E$ correspond to the points $A'$, $B'$, $C'$, $D'$, $E'$ and the lines $k$ and $l$ to the lines $k'$ and $l'$.*

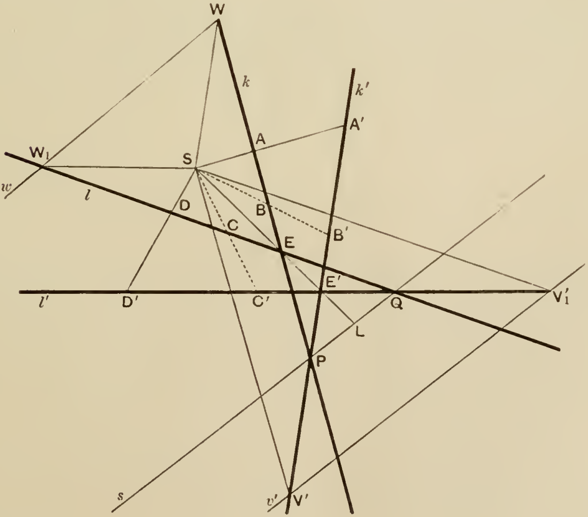
Let $k$ and $k'$ meet in $P$ and $l$ and $l'$ in $Q$. Let $PQ$ be $s$. Then it will be shown that the perspective which has $S$ for centre, $s$ for axis, and $E$ and $E'$ for corresponding points meets the requirements of the case.

(1) The lines $k$ and $k'$ which meet on $s$ and pass through $E$ and $E'$ are corresponding lines. Similarly $l$ and $l'$ are corresponding lines.

(2) Hence $A$ and $A'$; $B$ and $B'$; $C$ and $C'$; $D$ and $D'$ which are on these lines and are collinear with $S$ are corresponding points.

The point $G'$ corresponding to any point $G$ may be constructed as follows. Join $G$ to $S$ to meet $s$ in $H$. Then $G'$ is such that $(SHGG') = (SLEE')$ where $SEE'$ meets $s$ in $L$. (Art. 28.)

The line $g'$ corresponding to any given line $g$ may be constructed by joining $gs$ to $S$ by a line $h$ and taking $g'$ such that $(shgg') = (slkk')$ where $l$ is the connector of $kk'$ to $S$. (Art. 28.)



In connexion with the above it is seen that

(1) if a line $SFF'$ meet $k$ and $k'$ in $F$ and $F'$, the points $F$ and $F'$ may be determined one from the other as corresponding points of the ranges determined by $EBA$ and $E'B'A'$ respectively;

(2) if $W$ and $V'$ are the vanishing points of these ranges, parallels through $W$ and $V'$ to $k'$ and $k$ will intersect in $S$;

(3) if $W_1$ and $V_1'$ are the vanishing points of the ranges on $l$ and $l'$ parallels through $W_1$ and $V_1'$ to $l'$ and $l$ will intersect in $S$;

(4) the three lines $s$, $WW_1$ ($w$) and $V'V_1'$ ($v$) are parallel.

### 65. To place any two given Quadrangles $ABCD$ and $A'B'C'D'$ in Plane Perspective. (Figure, page 113.)

Let $AB$ be $k$; $A'B'$, $k'$; $CD$, $l$ and $C'D'$, $l'$. Let $kl$ and $k'l'$ be $E$ and $E'$. Regarding $EBA$ and $E'B'A'$ as determining two projective ranges on $k$ and $k'$ find the vanishing points $W$ and $V'$.

Similarly, regarding $ECD$ and $E'C'D'$ as determining two projective ranges on $l$ and $l'$ find the vanishing points $W_1$ and $V_1'$.

Place the second figure so that $V'V_1'$ is parallel to $WW_1$. Through $W$ and $W_1$ draw parallels to $k'$ and $l'$ meeting in $S$. Through $V'$ and $V_1'$ draw parallels to $k$ and $l$ meeting at $S'$. Keeping $WW_1$ and $V'V_1'$ parallel move the second figure till $S'$ coincides with $S$.

Consider the two quadrangles $SV'E'V_1'$ and $EWSW_1$. Five pairs of the lines joining these points are parallel, viz.:

| | |
|---|---|
| $SV'$ | $EW$, |
| $V'E'$ | $WS$, |
| $E'V_1'$ | $SW_1$, |
| $V_1'S$ | $W_1E$, |
| $V'V_1'$ | $WW_1$. |

Hence the lines joining the remaining pair of points are parallel, *i.e.* $SE$ and $SE'$. Therefore $S, E, E'$ are collinear. (Art. 13 (*b*), page 21.)

In the ranges on $k$ and $k'$ three pairs of corresponding points are collinear with $S$, viz.: $EE'$, $W\infty'$ and $V'\infty$. Hence the ranges are in perspective at $S$. Similarly, the ranges on $l$ and $l'$ are in perspective.

Hence by the last article the quadrangles are in perspective.

The above affords a solution of the following problems:

*Draw two transversals on which the sides of two given quadrangles determine equal corresponding segments.*

*Determine two points at which the corresponding vertices of two given quadrilaterals determine equal angles.*

## 66. Properties of Figures projective or in Plane Perspective.

The figures constructed from $ABCD$ and $A'B'C'D'$ in Art. 64 are in perspective. If one of them be displaced in its plane the two figures represent the most general case of two figures which are projective for they comply with the conditions laid down in Case (3), Art. 62. In fact, the figures may be constructed without reference to $S$ or $s$. Corresponding points on $k$ and $k'$ may be constructed as corresponding points of the ranges determined by $EAB$ and $E'A'B'$. Similarly, pairs of corresponding points may be constructed on $l$ and $l'$. Any line in the first figure will meet $k$ and $l$ in a pair of points and the corresponding line is the connector of the corresponding points on $k'$ and $l'$. The point corresponding to a given point in one figure may be found as the point of intersection of a pair of lines corresponding to any pair of lines through the given point. Hence

> *Any two quadrangles determine two projectively related figures*, and conversely,

> *The relationship of any two projective figures is completely determined if two corresponding quadrangles are given.*

Hence from Art. 65 it is seen that

> *Any two projective figures may be superposed so as to be in plane perspective.*

Art. 65 also enables us to

*Project any straight line in a given figure into the line at infinity and any three points into a triangle of given dimensions.*

For in the figure $DA$ may be projected into the line at infinity and $CBE$ into any triangle.

If the points $A$, $B$, $C$, $D$ coincide with the points $A'$, $B'$, $C'$, $D'$ the two figures must completely coincide. Therefore

*If two projective figures have four non-collinear self-corresponding points the figures must completely coincide.*

*If two projective figures have three collinear self-corresponding points they must be in perspective.*

In Figure, page 113, suppose $E'$, $B'$, $A'$ to coincide with $E$, $B$, $A$ respectively. Then $k'$ coincides with $k$ and every point on this line corresponds

8—2

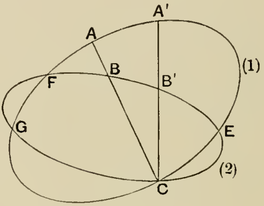
to itself. This line is the axis of perspective $s$. $S$ is determined as the point of intersection of the lines $DD'$ and $CC'$ joining two pairs of corresponding points on the two corresponding lines $l$ and $l'$.

**67.** *Two projective coplanar figures have in general three double or self-corresponding points.*

This proof involves the definition of a conic given, Art. 92.

If $A$, $A'$ are a pair of fixed corresponding points and $P$, $P'$ be any other pair of corresponding points, $AP$ and $A'P'$ are a pair of corresponding rays of two pencils, and, for different positions of $P$ and $P'$, the locus of $Q$ their point of intersection is a conic (1) through $A$ and $A'$ and every point on this conic is such that, if it be joined to $A$ and $A'$, the rays so formed are corresponding lines of the two figures.

Take $AA'$ and $BB'$ two fixed pairs of corresponding points and let $AB$ and $A'B'$ meet at $C$. Generally $C$ is not a self-corresponding point. Describe by the method explained above the two conics (1) and (2) through $AA'$ and through $BB'$ respectively, which are such that the lines obtained by joining points on them to $A$, $A'$ and $B$, $B'$ are corresponding lines. These conics will intersect in $C$ and in three other points $E$, $F$ and $G$. Since the lines $AE$, $BE$ correspond to $A'E$ and $B'E$, the point $E$ in the two figures corresponds to itself. Similarly $F$ and $G$ are self-corresponding points. Hence there are three self-corresponding or double points.



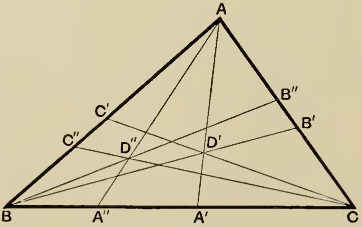
Two of the three points $E$, $F$, $G$ may be a pair of conjugate imaginary points.

*Particular case of the preceding.*

If the conic (1) break up into a pair of straight lines $AA'$ and $s$, let $AA'$ and $PP'$ meet at $S$. Then $AP$ and $A'P'$ will meet at some point $U$ on $s$. Let $R$ be any point on $AP$ and $R'$ the corresponding point, which will lie on $A'P'$. Hence since $(APRU)=(A'P'R'U)$, $RR'$ will pass through $S$. Also every point on $s$ will correspond to itself and therefore every pair of corresponding lines must intersect on $s$. Hence by Art. 62 the figures are in plane perspective $S$ and $s$ being the centre and axis of perspective.

*Analytical relation between two superposed projective figures.*

Let $A$, $B$, $C$ be the three self-corresponding points of the two figures and let $D'$ and $D''$ be two corresponding points. If $AD'$, $BD'$, $CD'$ meet the opposite sides of $ABC$ in $A'$, $B'$, $C'$, the ratios of $A'$, $B'$, $C'$ determine the position of $D'$ and may be termed its coordinates.



Let $x_0'$, $y_0'$, $z_0'$ be the coordinates of $D'$ referred to $ABC$, and $x_0''$, $y_0''$, $z_0''$ the coordinates of $D''$ also referred to $ABC$.

Let $P'$ and $P''$ be any two other corresponding points of the figures whose coordinates are $x'y'z'$ and $x''y''z''$.

Then $$\frac{x'}{x_0'} = (A \,.\, BCP'D') = (A \,.\, BCP''D'') = \frac{x''}{x_0''},$$

$$\therefore\ x'' = x' \frac{x_0''}{x_0'} = x' \,.\, k \quad \text{(suppose)},$$

so $$y'' = y' \frac{y_0''}{y_0'} = y' \,.\, l \qquad ,,$$

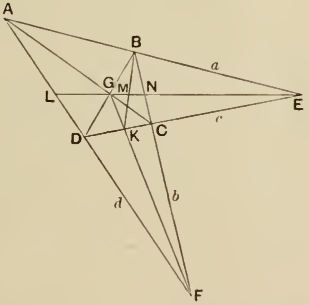
$$z'' = z' \frac{z_0''}{z_0'} = z' \,.\, m \qquad ,,$$

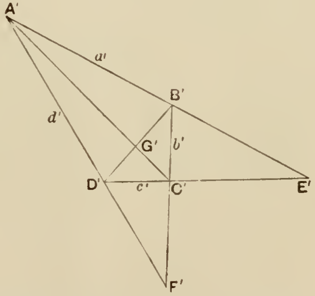
where $$klm = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

Hence the equation of the projection of any curve can be found at once. By varying $k$, $l$, $m$ subject to condition (i) different projective figures are obtained.

### 68. The Quadrangle.

Certain properties of a quadrangle follow at once from the fact that any quadrangle (or quadrilateral) is projective with any other quadrangle (or quadrilateral), Art. 65 and Art. 27.





(*a*) Consider any two quadrangles $ABCD$ and $A'B'C'D'$ or any two quadrilaterals $abcd$ and $a'b'c'd'$.

These are projective.

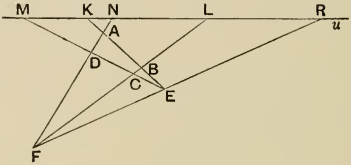
Hence any range or pencil in one figure must have the same anharmonic ratio as the corresponding range (or pencil) in the other figure. But the second figure may be looked upon as any quadrangle or quadrilateral. Hence the value of the anharmonic ratio in question must be independent of the relative positions of $A$, $B$, $C$, $D$ or those of $a$, $b$, $c$, $d$.

Thus in the figure the anharmonic ratio of the range $DCKE$ must not depend on the positions of $A$, $B$, $C$, $D$; it is $-1$.

Join $K$ to $B$ to meet $GE$ in $M$ and let $EG$ meet $d$ and $b$ in $L$ and $N$. Then $(LGME)$ must have some definite numerical value. This may be proved as follows:

$$(LGME)=1-(EGML)=1-\frac{(EGMN)}{(EGLN)}$$
$$=1-\frac{1-(EMGN)}{(EGLN)}=1-\frac{1-(-1)}{(-1)}=3.$$

Hence *the anharmonic ratio of any range or pencil derived by point and line construction from four points or four lines is independent of the position of these points and lines.*



(*b*) If any two points $N$ and $L$ be taken on any pair of sides $AD$ and $BC$ of a quadrangle $ABCD$, and points $N'$, $L'$ on the sides $A'D'$ and $B'C'$ of any other quadrangle $A'B'C'D'$, such that

$$(ADFN)=(A'D'F'N')$$

and $$(BCFL)=(B'C'F'L'),$$

and the lines $NL$ and $N'L'$ be $u$ and $u'$, then the quadrangle $ABCD$ and the line $u$ are projective with the quadrangle $A'B'C'D'$ and the line $u'$.

Consequently if $u$ and $u'$ meet the other sides of the quadrangle in $M$, $K$, and $M'$, $K'$ respectively,

$$(ABEK)=(A'B'E'K'),\quad (DCEM)=(D'C'E'M'),$$

and $$(KLMN)=(K'L'M'N').$$

Also since $A'$, $B'$, $C'$, $D'$ may be any four points, all the anharmonic ratios of ranges in the figure must be uniquely expressible in terms of $(FDAN)$ and $(FCBL)$ the two anharmonic ratios, which determine the position of $u$.

Thus let $FE$ meet $u$ in $R$ and let $(FDAN)=m$ and $(FCBL)=n$.

Projecting from $E$ on $u$,

$$m=(FDAN)=(RMKN),$$
$$n=(FCBL)=(RMKL),$$

$$\therefore\ (RMLN)=\frac{m}{n}\quad\text{and}\quad (RKLN)=\frac{1-m}{1-n}.$$

But $$(LNKM)=\frac{(LNRM)}{(LNRK)}=\frac{(RMLN)}{(RKLN)}=\frac{m}{n}\frac{(1-n)}{(1-m)}.$$

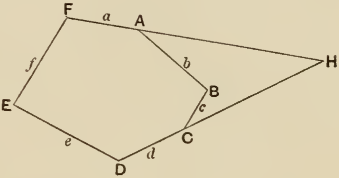
Projecting from $F$ on $u$,

$$(ECDM)=(RLNM)=\frac{1}{1-(RMLN)}=\frac{n}{n-m},$$

and $$(EBAK)=(RLNK)=\frac{1}{1-(RKLN)}=\frac{1-n}{m-n}.$$

In a similar manner, if another line be drawn to meet $AD$ and $CB$ in $N_1$ and $L_1$ and $(FDAN_1)$ and $(FCBL_1)$ be $m_1$ and $n_1$, the anharmonic ratios of all ranges and pencils in the resulting figure may be expressed in terms of $m$, $n$, $m_1$, and $n_1$.

(c) *If $a, b, c, d, e, f$ be the sides of a hexagon and three pairs of quadrilaterals $abcd, defa$; $bcde, efab$; and $cdef, fabc$ be formed out of the sides, then the points of intersection of the connectors of the middle points of the diagonals of these quadrilaterals taken in pairs are collinear.*



Let $ab, bc, cd, de, ef, fa$ and $ad$ be $A, B, C, D, E, F$ and $H$.

Then for a point $P$ inside the hexagon $ABCDEF$ on the connector of the middle point of the diagonals of $abcd$,

$$PAH - PBC = PHC - PAB$$

(Addendum 7),

and for a point $P$ on the connector of the middle points of the diagonals of $adef$,

$$PFH + PED = PHD + PEF \text{ (Addendum 7).}$$

If $P$ be the point of intersection of these lines, then

$$PFH - PAH + PED + PBC = PHD - PHC + PEF + PAB.$$

Therefore $$PFA + PED + PBC = PDC + PEF + PAB.$$

Hence the point of intersection is on the straight line (Addendum 7) which is the locus of points at which the sum of the areas subtended by the two groups of alternate sides of the hexagon are equal. The symmetry of the result proves that the two other points lie on the same line.

**69.** *To draw a straight line to cut five given straight lines in such a way that a point can be found at which the four segments subtend given angles.*

Let the five lines be $a, b, c, d, e$ and let $\alpha, \beta, \gamma, \delta$ be the angles to be subtended by the segments.

Take any point $P$ on $d$ and through $P$ draw a line $l$ to cut $abc$ in three points which with $P$ make a range of anharmonic ratio, the same as that of a pencil containing angles $\alpha, \beta, \gamma$. (Example 6, Chapter VI.)

Describe a conic to touch $abcd$ and $l$. (Art. 93.) All tangents to this conic cut $abcd$ in a range of the same anharmonic ratio as that in which it is cut by $l$. (Art. 93.)

Take any point $Q$ on $e$ and through $Q$ draw a line $m$ to cut $abc$ in three points which with $Q$ make a range of the same anharmonic ratio as that of the pencil containing the angles $\alpha, \beta, \gamma + \delta$.

Describe a conic to touch $abce$ and $m$. All tangents to this conic cut $abce$ in a range of the same anharmonic ratio as that in which it is cut by $m$.

Now the two conics have three common tangents $a, b, c$. Therefore they have a fourth common tangent $s$. Draw $s$. Let it be cut by $a, b, c, d, e$ in $A, B, C, D, E$. On $AB$ and $BC$ describe circles containing angles $\alpha$ and $\beta$. Let them intersect in $S$. Then $AB$ and $BC$ subtend angles $\alpha$ and $\beta$ at $S$.

Since the anharmonic ratio $ABCD$ is the same as that of a pencil $\alpha, \beta, \gamma$, $CD$ subtends an angle $\gamma$ at $S$. Similarly $CE$ subtends an angle $\gamma + \delta$ at $S$. Therefore $DE$ subtends an angle $\delta$ at $S$. Therefore $s$ and $S$ are the required line and point.

From the preceding the solution of the following problem can be obtained :

*To construct the perspective of four points so that it may be the same as the figure formed by four other given points.*

Let five of the lines joining the first four points be $a$, $b$, $c$, $d$, $e$, and those joining the corresponding points of the second set be $a'$, $b'$, $c'$, $d'$, $e'$. Let the angles between the latter be $\alpha$, $\beta$, $\gamma$, $\delta$.

Find $S$ and $s$ as in the preceding for the lines $a, b, c, d, e$ and the angles $\alpha, \beta, \gamma, \delta$. If $S$ and $s$ be taken as centre of perspective and vanishing line (Art. 23) the perspective figure will be similar to that formed by $a'b'c'd'e'$. By choosing the axis of perspective parallel to $S$ at a proper distance from $s$ the perspective figure can be made of the right linear dimensions (Art. 24).

**70.** The Correlative of Art. 64 is as follows :

*Given any two points $K$ and $L$ connected by a line $e$ and any two points $K'$ and $L'$ connected by $e'$, $s$ any line through $ee'$, and $aa'$, $bb'$, $cc'$, $dd'$ pairs of lines intersecting on $s$, $a$ and $b$ passing through $K$, $c$ and $d$ passing through $L$, $a'$ and $b'$ passing through $K'$, $c'$ and $d'$ passing through $L'$ ; then two figures can be constructed in plane perspective in which the lines $a$, $b$, $c$, $d$, $e$ correspond to the lines $a'$, $b'$, $c'$, $d'$, $e'$ and the points $K$ and $L$ to the points $K'$ and $L'$.*

Denote the lines $KK'$ by $p$ and $LL'$ by $q$. Let $pq$ be $S$. Then it will be shown that the perspective which has $s$ for axis and $S$ for centre and $e$ and $e'$ for corresponding lines meets our requirements.

(1) The points $K$ and $K'$ which are collinear with $S$ and lie on $e$ and $e'$ are corresponding points. Similarly $L$ and $L'$ are corresponding points.

(2) Hence $a$ and $a'$ ; $b$ and $b'$ ; $c$ and $c'$ ; $d$ and $d'$ which pass through these points and intersect on $s$ are corresponding lines.

The line corresponding to any line $g$ may be constructed as follows : Join $gs$ to $S$ by $h$. Then $g'$ is such that $(shgg') = (slee')$, where $l$ is the line joining the point $see'$ to $S$.

The point $G$ in one figure corresponding to the point $G'$ in the other may be determined as the point of intersection of two lines corresponding to two lines through $G'$.

## PROBLEMS OF THE FIRST DEGREE

**71.** Under this head are included theorems and problems:

(1) concerning loci the solutions of which are straight lines;

(2) the correlative problems in which the envelopes of straight lines are points;

(3) problems which have unique solutions.

The general method of solving questions coming under the heading (1) is to obtain the points of the required locus as the points of intersection of corresponding rays of two projective pencils. Then if it can be shown that the pencils are also in perspective the required locus is a straight line.

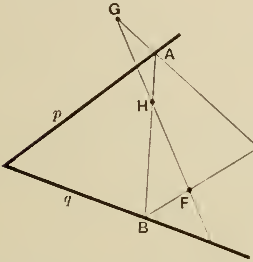
Problems coming under the heading (2) are treated by means of the correlative method. Two projective ranges are obtained and if it can be shown that they are also in perspective, it is known that the lines joining corresponding points all pass through a fixed point, which is the envelope of the lines joining pairs of corresponding points.

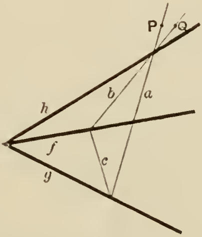
1. The vertices *A* and *B* of a triangle *ABC* move along two fixed lines *p* and *q* and the sides respectively pass through three fixed collinear points *H*, *G* and *F*.

Find the locus of *C*.

The sides *a* and *b* of a triangle *abc* pass through two fixed points *P* and *Q* and the vertices respectively move along three fixed concurrent lines *h*, *g* and *f*.

Find the envelope of *c*.





For different positions of the triangle *ABC*, the pencil formed by *FC* is projective with the range *B* and is therefore projective with the range *A* and with the pencil *GC*. Hence the pencils formed by *FC* and *GC* are projective. But if *FC* lies along *FG* then *GC* lies along *GF*. Hence *GF* is a self-corresponding ray of the two pencils and therefore the pencils are in perspective. Hence the locus of *C* is a straight line.

For different positions of the triangle *abc*, the range described by *fc* is projective with the pencil *b* and is therefore projective with the pencil *a* and with the range *gc*. Hence the ranges formed by *fc* and *gc* are projective. But if *fc* lies at *fg* then *gc* lies at *gf*. Hence *gf* is a self-corresponding point of the two ranges and therefore the ranges are in perspective. Hence the envelope of *c* is a point.

2. Find the locus of *C* in the above question if *G* and *F* are collinear with *pq* instead of with *H*.

As in the last question the pencils described by *FC* and *GC* are projective.

They are in perspective because the line joining *F*, *G*, *pq* is a self-corresponding ray.

Find the envelope of *c* in the above question if *g* and *f* are concurrent with *PQ* instead of with *h*.

As in the last question the ranges described by *fc* and *gc* are projective.

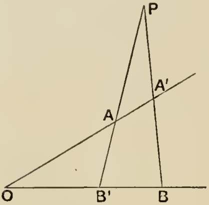
They are in perspective because the point of intersection of *f*, *g*, *PQ* is a self-corresponding point.

3. Two fixed points *A* and *B* are taken on two fixed lines which intersect

Two fixed lines *a* and *b* are taken passing through two fixed points situated at $O$ and on these lines variable points $A'$ and $B'$ are taken such that

$$OA+OB=OA'+OB'.$$

Find the locus of the point of intersection of $AB'$ and $BA'$.



Since $OA'+OB'$ is constant the distance $AA'$ equals the distance $BB'$ and therefore the ranges formed by $A'$ and $B'$ are projective. Hence the pencils $BA'$ and $AB'$ are projective.

When the point $A'$ coincides with $A$, the point $B$ coincides with $B'$, therefore the line $AB$ is a self-corresponding ray of the two pencils.

Therefore the pencils are in perspective and the locus of $P$ is a straight line.

4. In the preceding if

$$OA'-OB'=OA-OB$$

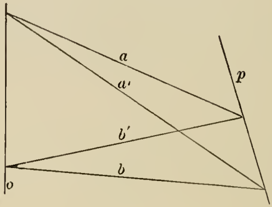
find the locus of the intersection of $AB'$ and $A'B$.

As in the previous case the pencils $AB'$ and $BA'$ are projective and in perspective.

on a line $o$ and through these points variable lines $a'$ and $b'$ are drawn such that

$$\widehat{oa}+\widehat{ob}=\widehat{oa'}+\widehat{ob'}.$$

Find the envelope of the lines connecting the points $ab'$ and $ba'$.



Since $\widehat{oa'}+\widehat{ob'}$ is constant the angle $\widehat{aa'}$ equals the angle $\widehat{bb'}$ and therefore the pencils formed by $a'$ and $b'$ are projective. Hence the ranges $ba'$ and $ab'$ are projective.

When the line $a'$ coincides with $a$, the ray $b$ coincides with $b'$, therefore the point $ab$ is a self-corresponding point of the two ranges.

Therefore the ranges are in perspective and the envelope of $p$ is a point.

In the preceding if

$$\widehat{oa'}-\widehat{ob'}=\widehat{oa}-\widehat{ob}$$

find the envelope of the line joining $ab'$ to $a'b$.

As in the previous case the ranges $ab'$ and $ba'$ are projective and in perspective.

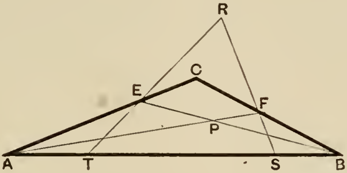
5. On the base $AB$ of a triangle $ABC$ two points $T$ and $S$ are taken such that $AT$ and $BS$ are in a given ratio. Through $T$ and $S$ lines are drawn to a fixed point $R$ so as to meet $AC$ and $BC$ in $E$ and $F$. Find the locus of the point of intersection of $EB$ and $FA$.

Because $AT=K.SB$ the ranges described by $T$ and $S$ are projective, and as these are projective, the ranges described by $E$ and $F$ are projective.

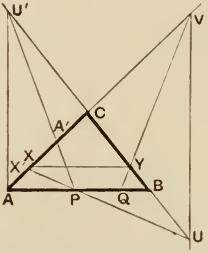
Hence the pencils $BE$ and $AF$ are projective.

When $T$ coincides with $A$, $S$ coincides with $B$ and $E$ and $F$ coincide with $A$ and

$B$. Hence the line $AB$ is a self-corresponding ray of the pencils described by $BE$ and $AF$. Therefore the pencils are in perspective and the locus of $P$ is a straight line.



6. $P$, $Q$ are given points on the base $AB$ of a triangle $ABC$. Find points $XY$ on the sides $AC$, $BC$ or their productions, such that if $PX$ meet $BC$ in $U$ and $QY$ meet $AC$ in $V$, $XY$ shall be parallel, and $UV$ perpendicular, to $AB$.



Take any point $X$ on $AC$. Draw $XY$ parallel to $AB$ to meet $CB$ in $Y$. Join $YQ$ meeting $AC$ in $V$. Draw $VU$ perpendicular to $AB$ to meet $CB$ in $U$. Join $UP$ to meet $AC$ in $X'$. Then the ranges $X$ and $X'$ are projective.

Take $X$ at $A$, $Y$ is then at $B$, $V$ is at $A$, $U$ is at $U'$, $X'$ is at $A'$. Take $X$ at $C$, $Y$ is then at $C$, $V$ is at $C$, $U$ is at $C$, $X'$ is at $C$.

Therefore $C$ is a self-corresponding point of the superposed ranges. The required point $Z$ the other self-corresponding point is found from the relation

$$(ACZX)=(A'CZX'),$$

whence

$$\frac{AZ}{CZ}:\frac{AX}{CX}=\frac{A'Z}{CZ}:\frac{A'X'}{CX'}$$

or

$$\frac{AZ}{A'Z}=\frac{AX}{CX}:\frac{A'X'}{CX'}.$$

# CHAPTER X

## PROJECTIVE FORMS IN RELATION TO THE CIRCLE:—ANHARMONIC PROPERTY. POLE AND POLAR. CIRCLES IN SELF-PERSPECTIVE

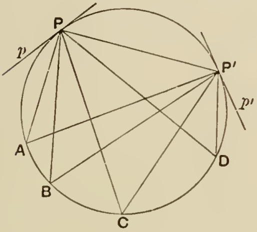
### 72. The Circle.

*Introductory.* A circle is usually defined as a curve such that the distance of every point on it from a fixed point is constant. It immediately follows (see Art. 73) that it is the locus of the points of intersection of corresponding rays of two directly equal pencils. In Art. 92 a conic is defined as the locus of the points of intersection of pairs of corresponding rays of two projective pencils. Hence a circle is a particular case of a conic. Also since equal pencils are projected into projective pencils, the projection of a circle is a conic and since the process of projection is reversible a conic may be projected into a circle. It would be possible to consider the projective properties of a conic and hence infer those of a circle. One method, however, of approaching the subject is to deduce by projection the properties of the conic from those of the circle.

In Art. 73 the anharmonic property of the circle is deduced from the usual definition. In the remainder of this chapter the properties of the circle are deduced solely from this anharmonic property. As this property is common to all conics the proofs (Arts. 74 to 81) hold equally for all conics. They have been stated only for the circle as it is thought that in this form they will be more easily grasped by the student in the first instance. In Chapter XI theorems and proofs special to the circle are given and in Chapter XIII and succeeding chapters the general case of the conic is considered.

## 73. The Anharmonic Property of the Circle.

*The anharmonic ratio of the pencil formed by joining four fixed points on a circle to a variable point on the circle is constant.*



Let $A$, $B$, $C$, $D$ be any four fixed points on the circle and $P$ and $P'$ any two positions of the variable point.

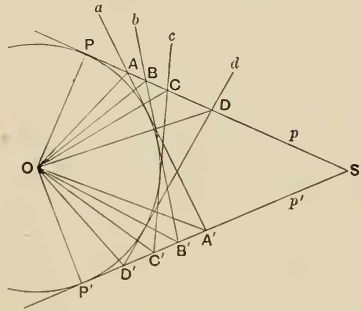
Then the angles $APB$ and $AP'B$ which stand on the same arc $AB$ are equal.

Similarly the angles $BPC$ and $BP'C$ are equal and likewise the angles $CPD$ and $CP'D$.

Hence the pencils $(P.ABCD)$ and $(P'.ABCD)$ contain equal angles between pairs of corresponding rays and are therefore equi-anharmonic and projective.

The tangent $p$ at $P$ corresponds to the ray $PP'$ of the pencil with vertex $P'$, and the tangent $p'$ at $P'$

*The anharmonic ratio of the range formed by the intersections of four fixed tangents to a circle with a variable tangent is constant.*



Let $a, b, c, d$ four fixed tangents to the circle meet two other tangents $p$ and $p'$ in $A$, $B$, $C$, $D$ and $A'$, $B'$, $C'$, $D'$ respectively.

Let $O$ be the centre of the circle. Then the angles $AOA'$ and $BOB'$ (Addendum 10) are equal and therefore the angles $AOB$ and $A'OB'$ are equal.

Similarly the angles $BOC$ and $B'OC'$ are equal and likewise the angles $COD$ and $C'OD'$.

Hence the ranges $(p.abcd)$ and $(p'.abcd)$ are such that corresponding segments subtend equal angles at $O$, and are therefore equi-anharmonic and projective.

The point of contact $P$ of the tangent $p$ corresponds to the point $pp'$ of the range on $p'$, and the corresponds to the ray $PP'$ of the pencil with vertex $P$.

Conversely:

*The locus of the points of intersection of corresponding rays of two directly equal pencils is a circle, which passes through the vertices of the two pencils.*

Let $PA$, $PB$ be the two rays of one pencil corresponding to the two rays $P'A$, $P'B$ of the other. Then since the angles $APB$ and $AP'B$ are equal, the circle through $A$, $P$, $P'$ also passes through $B$.

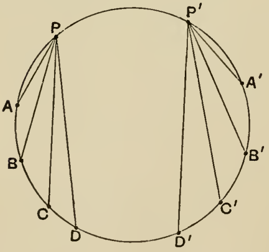
Similarly this circle passes through the point of intersection of any other pair of corresponding rays.

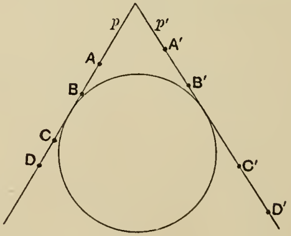
point of contact $P'$ of the tangent $p'$ corresponds to the point $pp'$ of the range on $p$.

*If two projective ranges on given bases are such that pairs of corresponding points subtend at a fixed point on a bisector of the angle between the bases an angle complemental to half the bisected angle, then the connectors of pairs of corresponding points envelope a circle of which the fixed point is the centre.*

Let $p$ and $p'$ be the bases of the ranges, $S$ their point of intersection, $O$ the fixed point, and $AA'$ any pair of corresponding points. Draw $OP$ and $OP'$ perpendiculars from $O$ to $p$ and $p'$. Then since $O$ is on the bisector of the angle $\widehat{pp'}$ a circle can be described with centre $O$ to touch $p$ and $p'$ at $P$ and $P'$. Also since the angle $AOA'$ equals the angle $POS$, the line $AA'$ touches this circle (Addendum 10). Similarly the line joining any other pair of corresponding points on $p$ and $p'$ touches this circle.

## 74. Projective Ranges on a circle, and Projective Systems of Tangents to a Circle.





If through the vertices $P$ and $P'$ of two projective pencils a circle be described meeting the rays of the pencils in $A$, $B$, $C$, $D$... and

If $ABCD$... and $A'B'C'D'$... be two projective ranges on bases $p$ and $p'$ and if tangents $a$, $b$, $c$, $d$... and $a'$, $b'$, $c'$, $d'$... be drawn from

$A'$, $B'$, $C'$, $D'$ … then these systems of points are such that if they are joined to any other points on the circle, the pencils so formed are projective. Such systems of points are termed *projective ranges on the circle.*

To any point considered as belonging to one range there corresponds one and only one point of the other.

If $A$, $B$, $C$, $D$ be four points on a circle, their anharmonic ratio, *i.e.* the anharmonic ratio of the pencil formed by joining them to any point on the circle, is denoted by $(ABCD)$.

these points to a circle which touches $p$ and $p'$, then these systems of tangents are such that if they are cut by any other tangents, the ranges so formed are projective. Such systems of tangents are termed *projective systems of tangents to the circle.*

To any tangent considered as belonging to one system of tangents there corresponds one and only one tangent of the other system.

If $a$, $b$, $c$, $d$ be four tangents to a circle, their anharmonic ratio, *i.e.* the anharmonic ratio of the range which they determine on any tangent to the circle, is denoted by $(abcd)$.

### Involutions of Points on, and of Tangents to, a Circle.

Similarly, if a circle be described through the vertex $S$ of an involution pencil to meet pairs of conjugate rays in $AA'$, $BB'$, $CC'$… these pairs of points are said to be conjugate points of an involution on the circle.

The pencil formed by joining four of these points to any point on the circle is projective with the pencil formed by joining their four conjugate points to any point on the circle, for each of these pencils is projective with the pencil formed by joining the same points to a point on the circle.

Two pairs of conjugate points

Similarly, if a circle be described to touch the base $s$ of an involution and the tangents to the circle from pairs of conjugate points be $aa'$, $bb'$, $cc'$… these pairs of tangents are said to be conjugate rays of an involution of tangents to the circle.

The range formed by the intersections of four of these tangents with any tangent to the circle is projective with the range formed by the intersections of the four conjugate tangents with any tangent, for each of these ranges is projective with the range formed by the intersections of these tangents with a tangent to the circle.

Two pairs of conjugate rays—

on the circle determine the involution and the conjugate of a given point in a given involution can be found from the fact that the range on the circle formed by any four points must be projective with the range formed by the conjugate points.

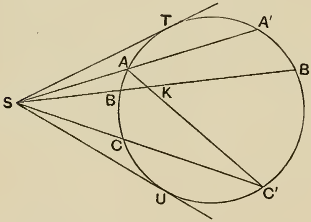
Conversely, if this condition is satisfied the points form an involution on the circle.

tangents to the circle—determine the involution and the conjugate of a given ray—tangent—in a given involution can be found from the fact that the pencil of tangents to the circle formed by any four tangents must be projective with the pencil formed by the conjugate rays—tangents.

Conversely, if this condition is satisfied the tangents form an involution of tangents to the circle.

### 75. Involution on a Circle and Involution of Tangents to a Circle.

*If through any point $S$ chords be drawn to intersect a circle in $AA'$, $BB'$, $CC'$... these pairs of points are pairs of conjugate points of an involution on the circle.*



Join $AC'$ to meet $BB'$ in $K$. Then

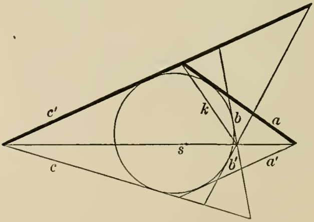
$$(BA'B'C')=(A\,.\,BA'B'C')$$
$$=(BSB'K)$$

by taking intersections of the pencil on $BB'$.

Similarly

$$(B'ABC)=(C'\,.\,B'ABC)$$
$$=(B'KBS).$$

*If from points on a line $s$ pairs of tangents $aa'$, $bb'$, $cc'$... be drawn to a circle, these pairs of tangents intersect any other tangent in pairs of conjugate points of an involution.*



Join $ac'$ to $bb'$ by $k$. Then

$$(ba'b'c')=(a\,.\,ba'b'c')$$
$$=(bsb'k)$$

by connecting points of the range to $bb'$.

Similarly

$$(b'abc)=(c'\,.\,b'abc)$$
$$=(b'kbs).$$

But $(BSB'K) = (B'KBS)$.

Therefore

$$(BA'B'C') = (B'ABC).$$

Therefore the three pairs of points $AA'$, $BB'$, $CC'$ are pairs of conjugate points of an involution on the circle.

If $S$ be a point external to the circle two tangents can be drawn from $S$ to touch the circle at $T$ and $U$. $T$ and $U$ are the double points of the involution determined by the two pairs of conjugate points $AA'$, $BB'$.

Since in this case

$$(BUB'T) = (B'UBT),$$

the points of contact $T$ and $U$ of the tangents from any point $S$ to the circle and the points of intersection $B$ and $B'$ of any chord through $S$ are four points on the circle which form a harmonic range, *i.e.* the pencil formed by joining these points to any point on the curve is harmonic.

Since the relation between pairs of harmonic conjugates is reciprocal, the tangents at $B$ and $B'$ intersect on the chord $TU$.

But $(bsb'k) = (b'kbs)$.

Therefore

$$(ba'b'c') = (b'abc).$$

Therefore the three pairs of points of intersection of the tangents $aa'$, $bb'$, $cc'$ with any tangent are pairs of conjugate points of an involution.

If the line $s$ meets the circle in two points, two tangents $t$ and $u$ can be drawn at these points. $t$ and $u$ are the double rays of the involution system of tangents determined by the pairs of conjugate rays $aa'$, $bb'$.

Since in this case

$$(bub't) = (b'ubt),$$

the tangents $t$ and $u$ at the points of intersection of a chord $s$ with the circle and the tangents $b$ and $b'$ from any point on $s$ are four tangents to the circle which form a harmonic system of tangents, *i.e.* a system of tangents which intersects any other tangent in a harmonic range.

Since the relation between pairs of harmonic conjugates is reciprocal, the chord of contact of the tangents $b$ and $b'$ passes through $tu$.

Hence the Fundamental Property of Harmonic Ranges of Points and Harmonic Pencils of Tangents to a Circle is obtained, viz.

*If through the point of intersection of the tangents at any points $T$ and $T'$ on a circle, a chord be drawn to meet the circle in $A$ and $A'$, then the range $TT'AA'$ is harmonic.*

*If from any point on the chord of contact of any two tangents $t$ and $t'$ to a circle, two other tangents $a$ and $a'$ be drawn to the circle, then the pencil of tangents $tt'aa'$ is harmonic.*

Conversely:

*If $AA'$, $BB'$, $CC'$ are pairs of conjugate points of an involution on a circle, the lines $AA'$, $BB'$, $CC''$ are concurrent.*

The involution is determined by two pairs of conjugate points $AA'$, $BB'$. Let $AA'$, $BB'$ meet at $S$. Join $C$ to $S$ to meet the circle in $C''$. Then $C''$ is a conjugate of $C$ in the given involution and must therefore coincide with $C'$.

Hence $CC'$ passes through $S$.

*If $aa'$, $bb'$, $cc'$ are pairs of conjugate rays of an involution pencil of tangents to a circle, the points $aa'$, $bb'$, $cc'$ are collinear.*

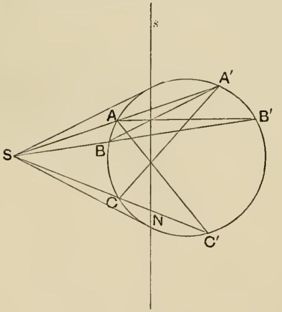
The involution is determined by two pairs of conjugate tangents $aa'$, $bb'$. Let the connector of $aa'$ to $bb'$ be $s$. Let $c$ meet $s$ in $C$ and let $c''$ be the tangent from $C$ to the circle. Then $c''$ is a conjugate of $c$ in the given involution and must therefore coincide with $c'$.

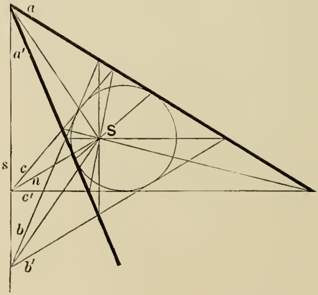
Hence $cc'$ lies on $s$.

### 76. Pole and Polar.

*If through any fixed point a variable chord of a circle be drawn, the locus of the harmonic conjugates of the fixed point with regard to the points of intersection of the variable chord with the circle is a straight line.*

*If from a variable point on a fixed straight line pairs of tangents be drawn to a circle, the envelope of the harmonic conjugate of the fixed line with regard to the pair of variable tangents is a fixed point.*





Draw through $S$ three chords meeting the circle in $AA'$, $BB'$, $CC'$. The first two chords may be regarded as fixed and the last as variable. The six points form an involution on the circle and therefore the pencil formed by joining $A$, $A'$, $B$, $B'$, $C$, $C'$ to $A'$ is

From any three points on $s$ draw pairs of tangents to the circle $aa'$, $bb'$, $cc'$. The first two pairs may be regarded as fixed and the last as variable. The six tangents form an involution of tangents and therefore the range formed by the intersections of $a'$

projective with the pencil formed by joining the conjugate points $A'$, $A$, $B'$, $B$, $C'$, $C$ to $A$.

Since these pencils have a common self-corresponding ray in $AA'$ they are also in perspective. (Art. 34.)

Therefore the corresponding rays intersect on a line $s$. This line is a fixed line since it passes through the points $AB' . BA'$ and $AB . A'B'$.

By the harmonic property of the quadrangle, $s$ meets $AA'$, $BB'$, $CC'$ in points which are harmonic conjugates of $S$ with regard to $AA'$, $BB'$, $CC'$ respectively.

Hence, if $CC'$ meets $s$ in $N$, $N$ is the point whose locus is required and its locus is the straight line $s$.

If $CC'$ be taken to coincide with $AA'$ it is seen that the tangents at $A$ and $A'$ (and therefore those at $B$ and $B'$, and $C$ and $C'$) intersect on $s$.

The line $s$ is called the *polar* of $S$.

By comparison of the work in the two columns it will be seen that $S$ is the pole of $s$.

From the preceding the following construction is obtained.

*Construction of the polar of a given point.*

Through $P$ the given point draw two chords $PBB'$ and $PAA'$.

with $a, a', b, b', c, c'$ is projective with the range formed by the intersections of $a$ with the conjugate rays $a'$, $a$, $b'$, $b$, $c'$, $c$.

Since these ranges have a self-corresponding point in $aa'$ they are also in perspective. (Art. 34.)

Therefore the corresponding points are collinear with a point $S$. This point is a fixed point since it is the intersection of the lines $ab' . a'b$ and $ab . a'b'$.

By the harmonic property of the quadrilateral, the connectors of $S$ with $aa'$, $bb'$, $cc'$ are harmonic conjugates of $s$ with regard to $aa'$, $bb'$, $cc'$ respectively.

Hence, if $n$ be the connector of $cc'$ to $S$, $n$ is the line whose envelope is required and its envelope is the point $S$.

If $cc'$ be taken to coincide with $aa'$ it will be seen that the points of contact of $a$ and $a'$ (and therefore those of $b$ and $b'$, and $c$ and $c'$) are collinear with $S$.

The point $S$ is called the *pole* of $s$.

By comparison of the work in the two columns it will be seen that $s$ is the polar of $S$.

From the preceding the following construction is obtained.

*Construction of the pole of a given line.*

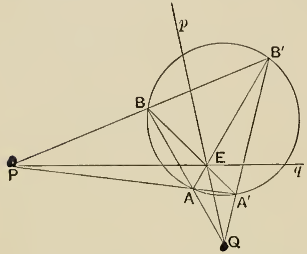
From two points on the given line $p$ draw pairs of tangents $aa'$ and $bb'$.

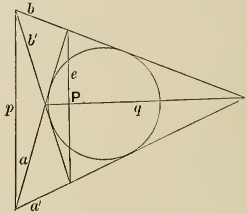
The polar $p$ of $P$ is the line joining $AB'.BA'$ (or $E$) to $AB.A'B'$ (or $Q$).

If $P$ is a point external to the circle, the polar may be constructed as the chord of contact of the tangents from $P$.

The pole $P$ of $p$ is the point of intersection of $ab'.ba'$ (or $q$) and $ab.a'b'$ (or $e$).

If $p$ meets the circle in real points the pole may be constructed as the point of intersection of the tangents at these points.





### Conjugate points and lines with respect to a circle.

If $P$ and $Q$ are two points such that the polar of one passes through the other, the points are called *conjugate points with respect to the circle.*

If the line joining two conjugate points $P$ and $Q$ meets the circle in real points these points are harmonic conjugates of $P$ and $Q$.

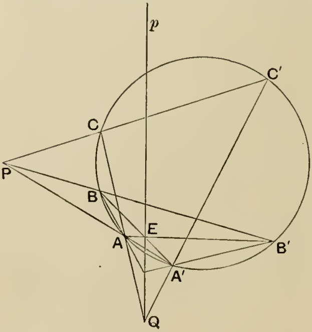
If $p$ and $q$ are two lines such that the pole of one lies on the other, the lines are called *conjugate lines with respect to the circle.*

If tangents can be drawn to the circle from the point of intersection of two conjugate lines $p$ and $q$ these tangents are harmonic conjugates of $p$ and $q$.

*If $p$ the polar of $P$ passes through $Q$, then $q$ the polar of $Q$ passes through $P$.*

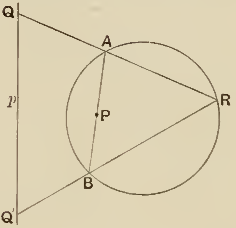
If $PQ$ meets the circle in real points $A$ and $A'$ the theorem is obvious since $PQAA'$ is harmonic. If $PQ$ does not meet the circle the theorem may be proved as follows:

Through $P$ draw chords $PAA'$, $PBB'$. Then $p$ is the line joining $BA'.AB'$ to $AB.A'B'$. Take $Q$ any point on $p$. Join $QA$ and $QA'$ to meet the circle in $C$ and $C'$. Then $CC'$ must pass through $P$. For if it does not let $PC$ meet the circle again in $C''$. Then $C''A'$ must pass



through $Q$ for $CA$ and $C''A'$ must intersect on $p$. Therefore $C'$ and $C''$ coincide. Hence $q$ the polar of $Q$ is the line joining $P$ to $CA'.AC'$ and therefore passes through $P$.

*Construction for pairs of points on a given straight line conjugate with respect to a circle.*

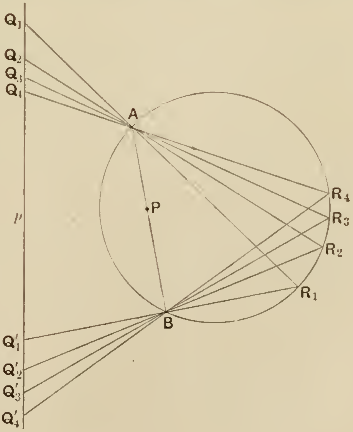


Let $p$ be the given line and $P$ its pole. Through $P$ draw any fixed chord $AB$. Let $Q$ be any point on $p$. To find its conjugate join $Q$ to $A$ to meet the circle in $R$. Let $RB$ meet $p$ in $Q'$. Then $Q$ and $Q'$ will be shown to be conjugate points.

Join $PR$ to meet the circle in $R'$. Then the polar of $P$ is the line joining $AR'.BR$ to $BR'.AR$. Therefore since $p$ is the polar of $P$, $AR'$ and $BR'$ must pass through $Q'$ and $Q$ respectively. Therefore $QP$ is the polar of $Q'$ and $PQ'$ is the polar of $Q$. Hence $Q$ and $Q'$ are conjugate points with respect to the circle.

Conversely it follows that, *if $Q$ and $Q'$, any two conjugate points with regard to a circle, be joined to any point $R$ on the circle by lines, which meet the circle in $A$ and $B$, then the line $AB$ passes through the pole of the line $QQ'$.*

**77.** *A circle determines by means of pairs of collinear conjugate points a definite involution on every straight line in its plane.*



If four points $Q_1$, $Q_2$, $Q_3$, $Q_4$ be taken on any straight line $p$, their conjugates $Q_1'$, $Q_2'$, $Q_3'$, $Q_4'$ with respect to the circle may be constructed by means of the last Article. In the figure, the range $Q_1Q_2Q_3Q_4$ is projective with the range $R_1R_2R_3R_4$ on the circle and this is projective with the range $Q_1'Q_2'Q_3'Q_4'$.

But since $Q_1$ and $Q_1'$ are conjugate points and the polar of each passes through the other, they mutually correspond. Hence the two projective ranges form an involution.

If the involution has real double points these points are the points of intersection of the line with the circle.

The correlative of the preceding is as follows:

*A circle determines by means of pairs of concurrent conjugate lines a definite involution pencil at every point in its plane.*

If $p_1$, $p_2$, $p_3$, $p_4$ are any four straight lines passing through any point $S$, their poles $P_1'$, $P_2'$, $P_3'$, $P_4'$ lie on $s$ the polar of $S$. Let $s$ meet $p_1, p_2, p_3, p_4$ in $P_1$, $P_2$, $P_3$, $P_4$. Then $P_1P_1'$, $P_2P_2'$, $P_3P_3'$, $P_4P_4'$, ... form an involution of conjugate points. Hence if $p_1'$, $p_2'$, $p_3'$, $p_4'$ be the lines joining $P_1'$, $P_2'$, $P_3'$, $P_4'$ to $S$, the pairs of conjugate lines $p_1p_1'$, $p_2p_2'$, $p_3p_3'$, $p_4p_4'$, ... are pairs of conjugate rays of an involution pencil.

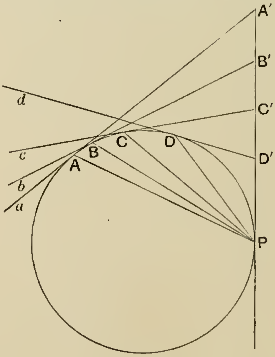
Since the tangents from $S$ to the circle, if they are real, are harmonic conjugates of each of these pairs of conjugate lines, they are the double rays of the involution.

*The anharmonic ratio of a range of four collinear points is equal to that of the pencil formed by their polars.*

If $Q_1$, $Q_2$, $Q_3$, $Q_4$ are four collinear points situated on a line $p$, their polars, which all pass through the pole of $p$, meet $p$ in points $Q_1'$, $Q_2'$, $Q_3'$, $Q_4'$, which are conjugates of $Q_1$, $Q_2$, $Q_3$, $Q_4$. Hence the range $Q_1Q_2Q_3Q_4$ is projective with the range $Q_1'Q_2'Q_3'Q_4'$ and therefore with the pencil formed by the polars of $Q_1$, $Q_2$: $Q_3$, $Q_4$.

*The anharmonic ratio of four points on a circle is equal to that of the four tangents at these points.*

Let $A$, $B$, $C$, $D$ be any four points on the circle and $a$, $b$, $c$, $d$ the tangents at these points. Let $P$ be any other point on the circle and let $p$ the tangent at this point meet $a$, $b$, $c$, $d$ in $A'$, $B'$, $C'$, $D'$.



Then the anharmonic ratio of $A$, $B$, $C$, $D$ is that of the pencil $(P\,.\,ABCD)$ and that of the tangents $a$, $b$, $c$, $d$ is $(A'B'C'D')$. Since $A'$, $B'$, $C'$, $D'$ are the poles of $PA$, $PB$, $PC$, $PD$, these anharmonic ratios are equal.

## 78. Properties of Pole and Polar.

The following is a restatement of the theorems of the last two Articles together with certain important deductions.

| | |
|---|---|
| (*a*) The line joining two conjugate points, if it meets the curve in real points, is divided harmonically by the curve.<br>For the polar of a point is the locus of points which are harmonic conjugates of that point with respect to the points of intersection of any chord through that point with the curve. | Two conjugate lines are harmonic conjugates of the two tangents, if real, from their point of intersection to the curve.<br>For the pole of a line is the envelope of lines which are harmonic conjugates of that line with respect to the tangents from any point on that line to the curve. |
| (*b*) The envelope of the polars of all points on a straight line is the pole of the line.<br>For the pole of a given line is on the polars of all points on the given line and therefore all the polars must contain the pole. | The locus of the poles of all lines through a given point is the polar of the point.<br>For the polar of a given point passes through the poles of all lines through the given point and therefore all the poles must lie on the polar. |
| (*c*) The pole of the connector of two points is the point of intersection of their polars. | The polar of the intersection of two lines is the connector of their poles. |
| (*d*) The polar of a point on the curve is the tangent at the point. | The pole of a tangent to the curve is its point of contact. |
| (*e*) Any point on a tangent is a conjugate of the point of contact of the tangent. | Any line through the point of contact of a tangent is a conjugate of the tangent. |
| (*f*) The polar of a point determines on the curve the points of contact—real or imaginary—of the tangents from the point to the curve.<br>For each point of contact is a conjugate of the point and therefore the polar passes through both points of contact. | The pole of a line determines the tangents—real or imaginary—to the curve at the points in which the line intersects the curve.<br>For each tangent is conjugate to the line and therefore passes through the pole of the line. |
| (*g*) The connectors of a pair of conjugate points to the pole of their connector are conjugate lines. | The intersections of a pair of conjugate lines with the polar of their point of intersection are conjugate points. |

(*h*) Two conjugate lines cut the curve in four points forming a harmonic range.

For each conjugate line passes through the pole of the other.

The tangents from two conjugate points to the curve form a harmonic pencil of tangents.

For each conjugate point lies on the polar of the other.

(*i*) The intersections of the polars of three given points form a triangle which is called *the polar*, or *conjugate triangle*, of the triangle formed by the three given points.

The connectors of the poles of three given lines form a triangle which is called *the polar*, or *conjugate triangle*, of the triangle formed by the three given lines.

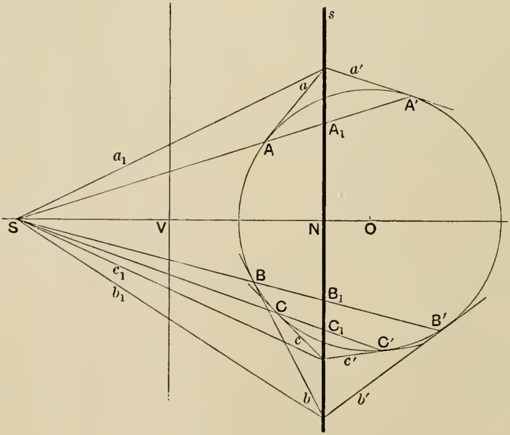
(*j*) A triangle such that each vertex is the pole of the opposite side is termed *a self-polar*, or *self-conjugate triangle.*

A triangle such that each side is the polar of the opposite vertex is termed *a self-polar*, or *self-conjugate triangle.*

### 79. Circle in Perspective with itself.

*Any circle may be looked upon as in harmonic perspective with itself, any point being the centre of perspective and the polar of that point the axis.*

*Any circle may be looked upon as in harmonic perspective with itself, any line being the axis of perspective and the pole of that line the centre.*



| | |
|---|---|
| Take any point $S$. Through $S$ draw chords $SAA_1A'$, $SBB_1B'$, $SCC_1C'$ to meet $s$ the polar of $S$ in $A_1$, $B_1$, $C_1$. Then the ranges $SA_1AA'$, $SB_1BB'$, $SC_1CC'$ are harmonic. | Take any line $s$. Draw pairs of tangents $aa'$, $bb'$, $cc'$ from points on $s$. Then, if $a_1$, $b_1$, $c_1$ are the lines joining $aa'$, $bb'$, $cc'$ to $S$ the pole of $s$, the pencils $sa_1aa'$, $sb_1bb'$, $sc_1cc'$ are harmonic. |
| Therefore, if $S$ be taken as centre, $s$ as axis of perspective, and the anharmonic ratio of the perspective be $-1$, $A$ corresponds to $A'$, $B$ to $B'$ and $C$ to $C'$ and vice versâ. Hence the circle is in self-perspective. | Therefore, if $s$ be taken as axis, $S$ as centre of perspective, and the anharmonic ratio of the perspective be $-1$, $a$ corresponds to $a'$, $b$ to $b'$ and $c$ to $c'$ and vice versâ. Hence the circle is in self-perspective. |

As a consequence, it is seen that

| | |
|---|---|
| (*a*) tangents at $A$ and $A'$, viz.: $a$ and $a'$, intersect on $s$. | the points of contact of tangents $a$ and $a'$, viz.: $A$ and $A'$, are collinear with $S$. |
| (*b*) corresponding chords $AB$ and $A'B'$ intersect on $s$. | corresponding points $ab$ and $a'b'$ are collinear with $S$. |
| (*c*) corresponding chords $AB'$ and $A'B$ intersect on $s$. | corresponding points $ab'$ and $a'b$ are collinear with $S$. |
| (*d*) tangents from $S$ touch the circle on $s$. | tangents where $s$ meets the circle intersect at $S$. |

From $S$ draw a line perpendicular to $s$ to meet $s$ in $N$. On $SN$ take a point $V$ such that $SV=VN$. Then the line parallel to $s$ through $V$ is the vanishing line.

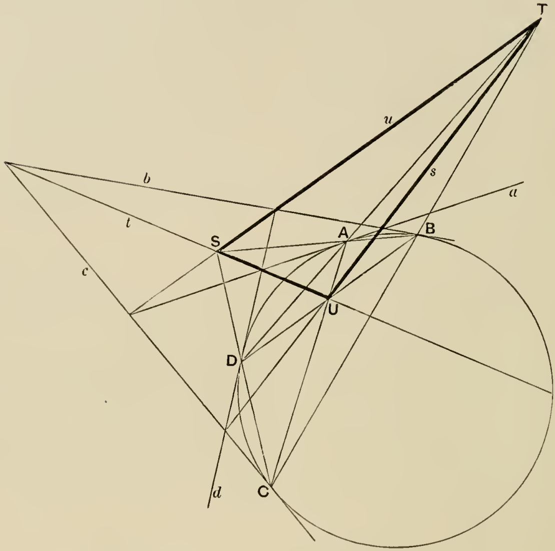
The two columns above simply give a restatement of the same theorem looked at from two slightly different points of view and form a good instance of the Principle of Duality.

### 80. Complete Inscribed Quadrangle and Complete Circumscribed Quadrilateral of a Circle.

| | |
|---|---|
| Any four points $A$, $B$, $C$, $D$ on a circle determine | Any four tangents $a$, $b$, $c$, $d$ to a circle determine |
| (1) three points $S$, $T$, $U$, as the points of intersection of their connectors taken in pairs. | (1) three lines $s$, $t$, $u$, as the lines joining the points of intersections of the tangents taken in pairs. |

(2) three lines $s$, $t$, $u$, as the connectors of the points $S$, $T$, $U$ in pairs.

(2) three points $S$, $T$, $U$, as the intersections of $s$, $t$, $u$ in pairs.



The lines $s$, $t$, $u$, are the polars of $S$, $T$, $U$ with regard to the circle. (Art. 76.) Hence the tangents $a$, $b$, $c$, $d$ at the points $A$, $B$, $C$, $D$ intersect in pairs on $s$, $t$, $u$.

As each side of the triangle $STU$ is the polar of the opposite vertex it is a self-polar or self-conjugate triangle.

The points $S$, $T$, $U$, are the poles of $s$, $t$, $u$. (Art. 76.) Hence the points of contact $A$, $B$, $C$, $D$ are collinear in pairs with $S$, $T$, $U$.

As each vertex of the triangle $stu$ is the pole of the opposite side it is a self-polar or self-conjugate triangle.

It follows that

(*a*) The diagonal points triangle $STU$ of the quadrangle $ABCD$ is the same as the diagonal triangle $stu$ of the quadrilateral $abcd$.

(*b*) From the quadrilateral *abcd*

$$\left.\begin{array}{llll} ac, & bd, & T, & S \\ ad, & bc, & U, & S \\ cd, & ba, & T, & U \end{array}\right\} \text{are harmonic ranges.}$$

(*c*) From the quadrangle *ABCD*

$$\left.\begin{array}{llll} AC.u, & BD.u, & S, & T \\ AD.t, & BC.t, & S, & U \\ DC.s, & AB.s, & T, & U \end{array}\right\} \text{are harmonic ranges.}$$

(*d*) If the circle meets $t$ in $P$, $P'$ and $s$ in $Q$, $Q'$

$$\left.\begin{array}{llll} P, & P', & S, & U \\ Q, & Q', & T, & U \end{array}\right\} \text{are harmonic ranges.}$$

In the figure, $u$ does not meet the circle in real points. One side of every self-conjugate triangle with regard to a circle does not meet the curve in real points.

## 81. Construction of a Self-Polar—or Self-Conjugate—Triangle of a Circle.

The triangle formed by two conjugate points and the pole of their connector as vertices is a self-conjugate triangle.

Given one vertex of such a triangle the number of triangles is infinite, for any two conjugate points on its polar may be taken as the remaining two vertices.

Given two vertices, which must be conjugate points, the triangle is completely determined, for the third vertex must be the pole of their connector.

The triangle formed by two conjugate lines and the polar of their point of intersection as sides is a self-conjugate triangle.

Given one side of such a triangle the number of triangles is infinite, for any two conjugate lines through its pole may be taken as the remaining two sides.

Given two sides, which must be conjugate lines, the triangle is completely determined, for the third side must be the polar of their point of intersection.

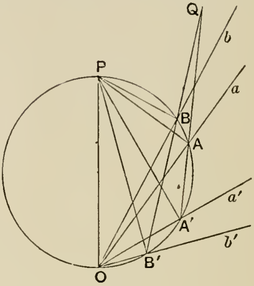
The limiting form of a self-polar *triangle*, if one vertex is given, *is the tangent from that point to the circle.* If the two conjugate points (which are vertices) are collinear with the centre of the circle, one of the vertices of the triangle is at infinity, and two sides are parallel. Every self-conjugate triangle with respect to a circle must be obtuse angled.

## EXAMPLES.

(1) If from a fixed point perpendiculars be drawn to the pairs of conjugate rays of a pencil in involution, the lines joining their feet taken in pairs pass through a fixed point.

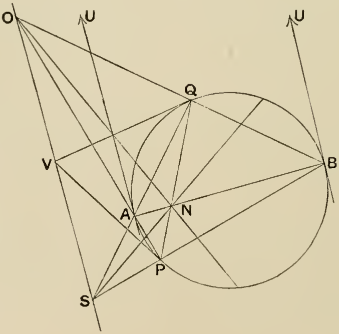
Let $aa'$, $bb'$ be conjugate rays of the involution pencil whose vertex is $O$ and $P$ any fixed point. A circle described on $OP$ as diameter will pass through $A$, $A'$, $B$, $B'$ the feet of the perpendiculars from $P$.

$P.AA'BB'\ldots$ is an involution pencil, since its rays are perpendicular to $a$, $a'$, $b$, $b'\ldots$. Hence $AA'$, $BB'\ldots$ are conjugate points of an involution on the circle. Therefore (Art. 75) $AA'$, $BB'$ ... pass through a fixed point $Q$.



(2) A fixed point $O$ external to a given circle is joined to the extremities $A$ and $B$ of any diameter, and $OA$, $OB$ meet the circle again in $P$, $Q$. Show that the tangents at $P$, $Q$ intersect on a fixed line parallel to the polar of $O$.

Consider the inscribed quadrangle $AQBP$ and the circumscribed quadrilateral formed by the tangents at these points. In the figure the points $U$, $V$, $O$, $S$ are collinear and form a harmonic range; therefore, since $U$ is at infinity, $V$ bisects $OS$. Therefore $V$ describes a line parallel to $NS$ which being the polar of $O$ is fixed.



(3) $A$, $B$, $C$ being three given points on a circle and $M$ any other point on the circle, $AM$, $BM$ are produced to meet the tangent at $C$ in $P$, $Q$, respectively; show that the difference of the reciprocals of $CP$, $CQ$ is constant.

Through $A$ and $B$ draw lines parallel to the tangent at $C$ to meet the circle in $A'$ and $B'$. Let $AB'$ and $A'B$ meet the tangent at $C$ in $O$ and $O'$. Then $O$ and $O'$ correspond respectively to the points at infinity in the projective ranges described by $P$ and $Q$, and $C$ is a self-corresponding point.

$$\therefore\ OP \,.\, O'Q = K^2 = OC \,.\, O'C,$$

$$\therefore\ (CP - CO)(CQ - CO') = OC \,.\, O'C,$$

$$\therefore\ CP \,.\, CQ = CO \,.\, (CQ - CP) \text{ since } CO' = -CO,$$

$$\therefore\ \frac{1}{CO} = \frac{1}{CP} - \frac{1}{CQ}.$$

(4) If a variable chord $CD$ of a circle passes through the middle point $O$ of a fixed chord $AB$ then the locus of the intersections of the lines $AD$, $CB$, is a straight line parallel to $AB$.

(5) If a variable chord $PQ$ of a circle passes through a fixed point $L$ and perpendiculars $PM$, $QN$ are drawn to the polar of $L$, then $\frac{1}{PM} + \frac{1}{QN}$ is constant.

(6) The polar of $O$ meets the circle in $P$, $Q$ and $PR$ is any chord of the circle through $P$. Through $O$ a line is drawn parallel to $PR$ meeting the circle in $U$, $V$. Show that $QR$ bisects $UV$.

Let $M$ be the point $QR \,.\, UV$. The range $PQUV$ on the circle is harmonic,

$$\therefore\ (R \,.\, PQUV) = -1,$$

$$\therefore\ (\infty MUV) = -1,$$

$$\therefore\ M \text{ bisects } UV.$$

# CHAPTER XI

## SPECIAL PROPERTIES OF THE CIRCLE:—INVERSE POINTS. COAXAL CIRCLES. CIRCLES IN PERSPECTIVE WITH EACH OTHER

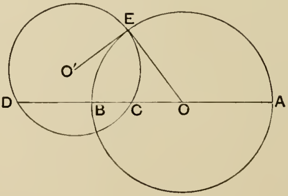
### 82. Inverse Points with respect to a Circle.

The theorems which have hitherto been proved for the circle are common to all conics and the same proofs hold for the conic in general as for the circle. Some of these theorems may be proved for the circle in a manner not applicable to the general case, and there are certain theorems true for the circle which do not hold for the conic. The more important of these are given in this chapter.

**Inverse Points.** Definition: *Two conjugate points collinear with the centre of a circle are termed inverse points.*

*If two circles cut each other orthogonally, each determines inverse points upon every diameter of the other*, and conversely,

*If one circle pass through inverse points with respect to another circle, the two circles cut orthogonally.*



Let $O$ be the centre of one circle and let a diameter of this circle meet it in $A, B$ and the second circle in $C, D$.

In the first case, since the circles are orthogonal, the line $OE$ joining $O$ to either of the points of intersection of the circles is a tangent to the second circle, and therefore

$$OC \,.\, OD = OE^2 = OA^2.$$

Therefore $(ABCD)$ is harmonic and $C$ and $D$ are inverse points with respect to the circle whose centre is $O$.

In the second case, since $C$ and $D$ are a pair of inverse points with respect to the circle whose centre is $O$,

$$OC \,.\, OD = OB^2 = OE^2.$$

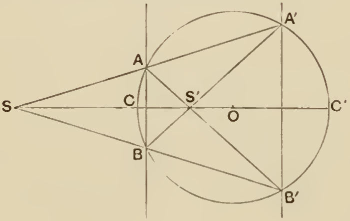
Therefore $OE$ is a tangent to the second circle and the two circles cut orthogonally.

*Cor.* I. To describe a circle through two given points to cut a given circle orthogonally. Let the given points be $A$ and $B$. Take $C$ the inverse of $A$ or of $B$ with respect to the circle. Then $ABC$ is the orthogonal circle.

*Cor.* II. To describe a circle through a given point to cut two given circles orthogonally. Let the given point be $A$. Take $B$ and $C$ the inverse points of $A$ with respect to the two circles. Then $ABC$ is the required orthogonal circle.

*Cor.* III. If $C$ and $D$ be conjugate points with respect to a circle ($X$), and a circle ($Y$) be described on $CD$ as diameter, this circle ($Y$) will cut the circle ($X$) orthogonally. For let $CD$ meet ($X$) in $A$ and $B$. Then ($CDAB$) is harmonic and since $CD$ is a diameter of ($Y$) $A$ and $B$ are inverse points with respect to ($Y$).

**Special Construction of Pole and Polar for a Circle.**



Let $O$ be the centre of the circle and $S$ the point whose polar is required. Join $OS$ to meet the circle at $C$ and $C'$ and draw two chords $SAA'$ and $SBB'$ equally inclined to $SO$.

Then by symmetry $AB'$ and $BA'$ will intersect at some point $S'$ on $SO$ and $AB$ and $A'B'$ will be perpendicular to $SO$.

Hence $s$ the polar of $S$ will be the line through $S'$ perpendicular to $SO$, Art. 76. Also because ($SS'CC'$) is harmonic and $O$ is the middle point of $CC'$,

$$OS \cdot OS' = OC^2,$$

while the points $S$, $S'$, being conjugate points collinear with the centre, are inverse points.

Hence *the polar of a given point with respect to a circle is the line through its inverse point perpendicular to the connector of the given point to the centre.*

Also:

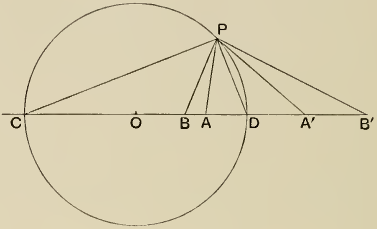
*The pole of a given line is the inverse point of the foot of the perpendicular from the centre on the given line.*

*If $AA'$, $BB'$ be two pairs of collinear inverse points, with respect to a circle, and $P$ be any point on the circle, then*

(1) *the ratio $AP : A'P$ is constant for different positions of $P$;*

(2) *the angles $APB$ and $A'PB'$ are equal or supplemental.* (Cf. Art. 53.)

Let the line $BAA'B'$ meet the circle at $C$ and $D$. Take $P$ any point on the circle. Then the ranges $CDAA'$ and $CDBB'$ are harmonic. Therefore the pencils $(P . CDAA')$ and $(P.CDBB')$ are harmonic. But $CP$, $PD$ are at right angles, and are therefore common bisectors of the angles $APA'$ and $BPB'$.



Hence (1) $AP : A'P = AD : A'D =$ a constant,

(2) the angles $APB$ and $A'PB'$ are equal or supplemental.

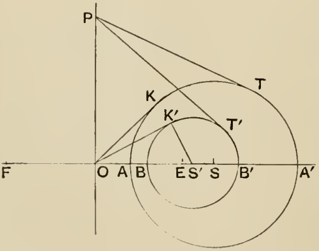
### 83. Coaxal Circles.

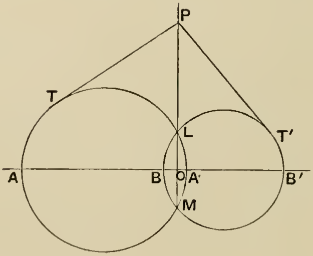
*If $AA'$, $BB'$, $CC'$, ... are pairs of conjugate points of an involution, the circles described on $AA'$, $BB'$, $CC'$, ... as diameters are said to form a coaxal system of circles.*

There are two cases to be considered according as the involution has, or has not, real double points.

If the double elements are real points $E$ and $F$, let $O$ be the centre of the involution.

If the double elements are imaginary, let $O$ be the centre of the involution.





If $PO$ be a line through $O$ perpendicular to the base of the involution, this line is termed *the radical axis* of the circles and it will be shown that the tangents from any point $P$ on this line to the circles of the system are equal.

Since

$$OE^2 = OA \,.\, OA' = OB \,.\, OB',$$

the tangents, $OK$ and $OK'$, from $O$ to any two of the circles are equal. If $R$ and $R'$ be the radii of the circles, $S$ and $S'$ their centres and $PT$ and $PT'$ the tangents from a point $P$ on the radical axis, then

$$PT^2 = PS^2 - R^2 = PO^2 + OS^2 - R^2$$
$$= PO^2 + OK^2.$$

Similarly

$$PT'^2 = PO^2 + OK'^2.$$

Therefore $PT = PT'$.

All circles described through $E$ and $F$, which are inverse points with respect to all the circles of the system, will (Art. 82) cut every circle of the coaxal system at right angles.

The circles through $E$ and $F$ form a system of coaxal circles of the nature described on the right-hand side, $E$ and $F$ taking the place of $L$ and $M$.

If $PO$ be a line through $O$ perpendicular to the base of the involution, this line is termed *the radical axis* of the circles and it will be shown that the tangents (if real) from any point $P$ on this line to the circles of the system are equal.

Take two points $L$ and $M$ vertically above and below $O$ such that

$$OL^2 = OM^2 = OA \,.\, OA' = OB \,.\, OB'.$$

Then all circles of the system will pass through $L$ and $M$. Hence if $PT$ and $PT'$ be the tangents to two circles for a point $P$ on the radical axis,

$$PT^2 = PL \,.\, PM$$

and

$$PT'^2 = PL \,.\, PM.$$

Therefore $PT = PT'$.

All circles with regard to which $L$ and $M$ are inverse points (Art. 82) will cut all circles of the coaxal system at right angles. $L$ and $M$ will be common inverse points of such a system of circles.

The circles which have $L$ and $M$ for common inverse points form a coaxal system of the kind described on the left-hand side, $L$ and $M$ taking the place of $E$ and $F$.

The points $E$ and $F$ are termed the *common inverse points* or *the limiting points* of the system of coaxal circles.

The Radical Axis is sometimes defined as the locus of points from which the tangents to a system of coaxal circles are equal. If the circles intersect in real points it may be constructed as the line joining these points. If the circles do not intersect in real points, let $t$ and $t'$ be two common tangents to two of the circles which touch them at $P$, $Q$ and $P'$, $Q'$, respectively. Then, if $M$ and $M'$ are the middle points of $PQ$ and $P'Q'$, the radical axis is the line $MM'$. Two circles completely determine a coaxal system, for two pairs of conjugate points $AA'$ and $BB'$ completely determine an involution. A system of coaxal circles is also determined by a point—a limiting point—and a circle. The radical axis in this case is termed the radical axis of the point and the circle.

*The three radical axes of three circles taken in pairs are concurrent.*

Let the circles be denoted by (1), (2) and (3). Consider $P$ the point of intersection of the radical axes of (1) and (2), and of (2) and (3). At this point the tangent to (1) equals the tangent to (2) and therefore the tangent to (3). Hence the radical axis of (1) and (3) passes through $P$.

*To describe a circle to cut three given circles orthogonally.*

Let $S$ be the point of intersection of the three radical axes of the circles taken in pairs. Then if $S$ be external to the circles the tangents from $S$ to the three circles are equal and, if a circle be described with centre $S$ and radius equal to the length of these tangents, this circle will cut the three circles orthogonally. If $S$ be an internal point the radius of the circle is an imaginary quantity.

### 84. Coaxal Circles in Self-Perspective.

*Every circle of a coaxal system, with real limiting points, is in harmonic perspective with itself, either of the limiting points being the centre, and its polar the axis, of perspective.*

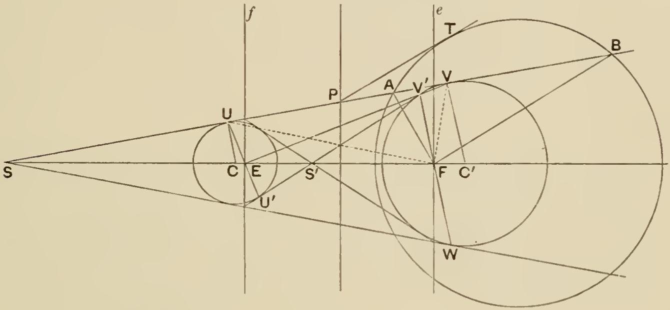
This theorem follows at once from Art. 79 when it is borne in mind that the limiting points have the same polars with respect to each circle of the coaxal system. The polar of each is the line through the other perpendicular to the line of centres.

The following are some of the more important properties of a system of coaxal circles with real limiting points. In these theorems the *centres of similitude of a pair of circles* occur. Generally a centre of similitude of two curves is a centre of perspective of the curves, which has the line at infinity as one of the corresponding axes of perspective (Art. 29). The centres of similitude of two circles are the two points, which divide the distance between the centres in the ratio of the radii of the circles (see example 22, page 159). When the common tangents to two circles are real, their points of intersection which lie on their line of centres are centres of similitude (Art. 87).

(1) If $U$ and $V$ be the points of contact of a common tangent to two of the circles of the system, the circle described on $UV$ as diameter cuts the circles orthogonally and will therefore pass through the limiting points $E$ and $F$, if these points are real. Hence the angles $UFV$ and $UEV$ are right angles.

(2) If $UV$ meets a circle of the system in $A$ and $B$, then $(UVAB)$ is harmonic. For if $UV$ meets the radical axis in $P$, then $PV^2=PT^2=PA\,.\,PB$, where $T$ is the point of contact of the tangent from $P$ to the circle $AB$.

Since $UFV$ is a right angle $FV$ bisects the angle $AFB$.



(3) If $UE$ and $VE$ meet the circles again in $U'$ and $V'$, then $U'V'$ is a common tangent to the circles and $UV$ and $U'V'$ intersect on $e$ the polar of $E$. This follows from the fact that the circles are in self-perspective with $E$ and $e$ as centre and axis of perspective.

(4) Hence $EF$, $f$, $e$ are the connectors of the points of intersection of the common tangents to the two circles and therefore form a common self-conjugate triangle. These lines are the same for every pair of circles of the coaxal system and therefore form a common self-conjugate triangle for the whole system of coaxal circles.

(5) Hence $S$ and $S'$, the points of intersection of pairs of common tangents to any pair of circles of the system which lie on the line of centres, are harmonic conjugates of $E$ and $F$. (Art. 80.)

(6) Hence the circle described on $SS'$ as diameter, *i.e.* the circle of similitude of any pair of circles of the system, is itself a circle of the coaxal system.

(7) If two chords through $E$ meet any pair of circles of the coaxal system in $A$, $A'$ and $B$, $B'$, then the points $AB\,.\,A'B'$ and $AB'\,.\,A'B$ are on the line $e$.

(8) If $U$, $V$, $U'$, $V'$ be joined to $C$ and $C'$, the centres of the circles, the joining lines are respectively perpendicular to $UV$ and $U'V'$.

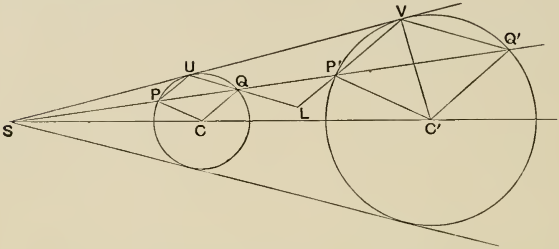
Hence if $r$ and $r'$ be the radii of the circles $\dfrac{CS'}{C'S'}=-\dfrac{r}{r'}=-\dfrac{CS}{C'S}$.

Hence the range $SS'CC'$ is harmonic. Therefore $S$ and $S'$ are the common harmonic conjugates of $EF$ and $CC'$.

(9) If a chord $SPQP'Q'$ be drawn to meet the circles in $P$, $Q$ and $P'$, $Q'$

$$\frac{SC}{CP}=\frac{SC'}{C'P'}.$$

Hence the triangles $SPC$ and $SP'C'$ are similar and $\dfrac{SP}{SP'}=\dfrac{r}{r'}$.



Similarly, $\dfrac{SU}{SV}=\dfrac{r}{r'}$. Hence from the triangles $SUP$ and $SVP'$ it is seen that the lines $PU$ and $P'V$ are parallel. Similarly the lines $UQ$ and $VQ'$ are parallel.

(10) The angle $SQU$ = angle $SQ'V$ = angle $UVP'$. Therefore $U$, $V$, $Q$, $P'$ are concyclic and if $UQ$ and $P'V$ meet at $L$, $LQ.LU=LP'.LV$. Hence the tangents from $L$ to the two circles are equal and $L$ is a point on the radical axis.

(11) Since the lines $UP$, $VP'$ are parallel, the pencils formed by joining points $P$ and $P'$ collinear with $S$ to $U$ and $V$ respectively are projective. Hence the ranges $P$ and $P'$ on the two circles are projective. Similarly, since $UQ$ and $VP'$ intersect on the radical axis, the ranges described by $Q$ and $P'$ on the circles are projective.

(12) The ratio of the tangents from any point on a circle of a coaxal system to two other given circles of the system is constant. For (Addendum 13) if $P$ be the point, $C$, $C_1$, $C_2$ the centres of the circles, $M$ and $N$ the points where the radical axis and the perpendicular from $P$ meet the line of centres and $PT_1$ and $PT_2$ the tangents, then

$$PT_1^2=2.MN.CC_1$$

and

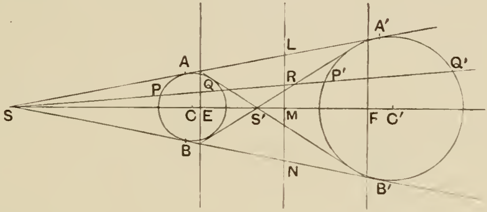
$$PT_2^2=2.MN.CC_2,$$

$$\therefore\ \frac{PT_1}{PT_2}=\sqrt{\frac{CC_1}{CC_2}}.$$

## 85. Two circles in Perspective with each other.

*Any two circles may be looked upon as in perspective with each other, either of the centres of similitude being the centre of perspective, and the axis of perspective either their radical axis or the line at infinity.*

In Art. 72 it was shown that the perspective of a circle is in general a conic. A conic is determined by five points, since it is the locus of points of intersection of corresponding rays of two projective pencils (Art. 92). If a perspective is such that five points on a circle correspond to five points on a conic, the conic and circle must be in perspective. Hence if in a perspective five points on one circle correspond to five points on another circle, the circles must be in perspective, since a circle is a particular form of a conic.



Given two triangles $APB$ and $A'P'B'$ such that the lines joining corresponding vertices meet at a point $S$, the two triangles determine a perspective of which the centre is $S$ and the axis the line joining the points of intersection of pairs of corresponding sides.

Let the tangents* from $S$, a centre of similitude of any two given circles, touch the circles in $A$, $B$ and $A'$, $B'$. Draw $SPQP'Q'$ any chord through $S$ to meet the circles in $P$, $Q$ and $P'$, $Q'$ respectively. Consider the perspective in which $AA'$, $BB'$, $PP'$ are pairs of corresponding points.

$S$ is the centre of perspective and the axis the line at infinity since the lines $AP$, $A'P'$; $PB$, $P'B'$; $AB$, $A'B'$ are parallel in pairs (Art. 84 (9)). In this perspective $A$, $B$, $A'$, $B'$ each count as two points on the circles and therefore five points on one circle correspond to five points on the other. Therefore the circles are in perspective.

Similarly, if the points $A$, $P$, $B$ are taken to correspond to the points $A'$, $Q'$, $B'$, a perspective is obtained in which $S$ is the centre of perspective and the radical axis is the axis of perspective, for (Art. 84 (10)) $AP$, $A'Q'$; $PB$, $Q'B'$; $AB$, $AB'$ intersect in pairs on the radical axis.

If the line at infinity is taken as the axis of perspective, the anharmonic ratio of the perspective is $(S\infty\, PP')$, which is constant.

Therefore $$\frac{SP}{SP'} = (S\infty\, PP') = (S\infty\, AA') = \frac{SA}{SA'} = \frac{r}{r'}.$$

In this case, both the vanishing lines are at infinity.

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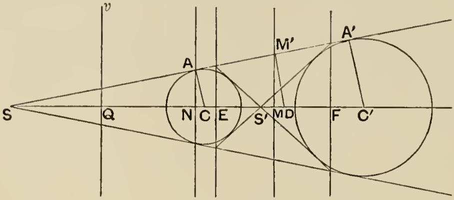
* The case when these tangents are imaginary is considered in example 22, page 159.

If the radical axis is taken as the axis of perspective, and the line $SPQ'$ meets the radical axis in $R$, $(SRPQ')$ is constant.



Hence if $M'$ be the middle point of $AA'$,

$$(SRPQ') = (SM'AA') = -\frac{SA}{SA'}, = -\frac{r}{r'}.$$

If the line of centres meets the vanishing line $v$ in the first figure in $Q$ and the radical axis in $M$, then

$$\frac{SQ}{MQ} = (SMQ\infty) = (SM'AA') = -\frac{r}{r'}.$$

Hence since $M'D$ in the figure equals $\frac{r+r'}{2}$,

$$\frac{2 . SQ}{SM} = \frac{r}{\frac{r+r'}{2}} = \frac{AC}{M'D} = \frac{SN}{SM}.$$

Therefore $SQ = QN.$

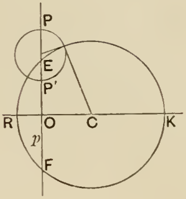
The line $v$ is the radical axis of the point $S$ and the circle, whose centre is $C$. It is half-way between $S$ and $N$ its inverse point with respect to the circle. The points $E$ and $F$ are corresponding points.

Similarly $S'$ may be taken as the centre of Perspective and the line at infinity or the radical axis as the axis of Perspective.

Hence it is seen that *if the perspective of a circle be formed with the radical axis of the circle and centre of perspective as vanishing line, the perspective is a circle having the axis of perspective for its radical axis with the given circle.*

### 86. Construction of the Involution determined on a given line by a circle.

In Art. 77 it was proved that, if $P$ be any point on a straight line $p$ and the polar of $P$ with respect to a circle meet this straight line in $P'$, then, for different positions of $P$, the points $P$ and $P'$ are pairs of conjugate points with respect to the circle and determine an involution on the given line $p$.

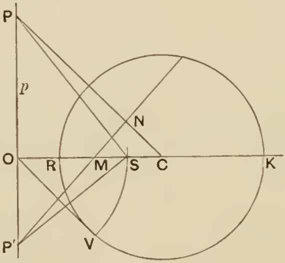


(*a*) If the line $p$ meet the circle in real points $E$ and $F$, these are the double points of the involution and $P$ and $P'$ are harmonic conjugates of these points. If $O$ be the foot of the perpendicular from $C$, the centre of the circle, on the line $p$, $O$ corresponds to the point at infinity in the involution and is the centre of the involution.

Therefore $OP \, . \, OP' = OE^2 = OF^2$.

A circle described on $PP'$ as diameter cuts the given circle at right angles (Art. 82). If $CO$ meet the circle in $R$ and $K$, then

$$OP \, . \, OP' = - OE \, . \, OF = - OR \, . \, OK.$$



(*b*) If the line $p$ meet the circle in imaginary points, and the perpendicular from $C$, the centre of the circle, meet the line in $O$, then $O$ is the centre of the involution and

$$OP \, . \, OP' = \text{a constant.}$$

Let the polar of $P$ meet $OC$ in $M$ and $CP$ in $N$; then since the triangles $OP'M$ and $OCP$ are similar

$$\frac{OM}{OP'} = - \frac{OP}{OC}.$$

$$\therefore \; OP \, . \, OP' = - OM \, . \, OC = - OR \, . \, OK.$$

But $OR \, . \, OK = OV^2$, where $OV$ is the tangent from $O$ to the circle.

Hence if a circle be described with centre $O$ and radius $OV$ to meet $OC$ at $S$, this circle cuts the given circle orthogonally, and the pairs of conjugate points $PP'$, $QQ'$ of the involution subtend right angles at $S$.

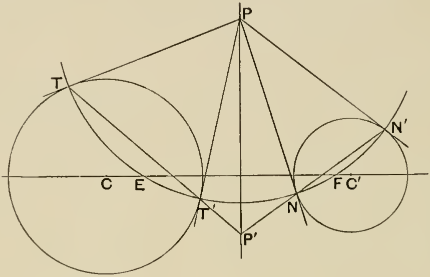
$S$ is one of the pairs of common harmonic conjugates of $MC$ and of $RK$.

### 87. Common Involution chords of a pair of circles.

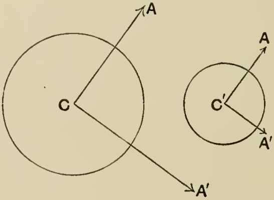
*Every pair of circles determines the same involution* (*a*) *on their radical axis and* (*b*) *on the line at infinity.*

(*a*) If the circles intersect in real points $L$ and $M$, the involution on the radical axis $LM$ in the case of each circle consists of pairs of points which are harmonic conjugates of $L$ and $M$, and therefore both circles determine the same involution on the radical axis.

If the circles do not intersect in real points, let $P$ be any point on their radical axis. The tangents from $P$ to the two circles are equal and a circle with centre $P$ and radius equal to these tangents intersects both the circles at right angles. Let $TT'$, $NN'$ be the chords of intersection of this orthogonal circle with the two given circles. Since the radical axes of the three circles taken in pairs are concurrent, the chords $TT'$, $NN'$ intersect at some point $P'$ on the radical axis of the two given circles. But $TT'$, $NN'$ are the polars of $P$ with respect to the given circles. Hence $P$ and $P'$ are conjugate points with respect to both circles and, since $P$ may be any point on the radical axis, both circles determine the same involution on their radical axis.



(*b*) Let $C$ and $C'$ be the centres of the two circles. Then $C$ and $C'$ are the poles of the line at infinity with respect to these circles and pairs of conjugate lines through $C$ and $C'$ meet the line at infinity in pairs of conjugate points of the involutions determined on that line by the two circles.



Draw any two lines at right angles through $C$, viz. $CA$ and $CA'$. These are a pair of conjugate lines with respect to the first circle.

Draw parallel lines through $C'$. These are conjugate lines with respect to the second circle. But parallel lines meet the line at infinity in the same points. Hence $CA$, $CA'$ and $C'A$, $C'A'$ determine the same pair of conjugate points on the line at infinity. Since $CA$ and $CA'$ may be any pair of conjugate lines, the involutions determined by the two circles on the line at infinity are the same. The imaginary double points of this involution are termed *the circular points at infinity.*

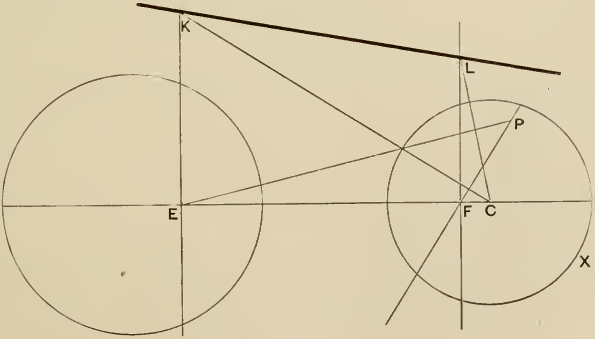
**88.** (*a*) *The envelope of the polars of a fixed point with respect to a system of coaxal circles is a point.*

Let $O$ be any fixed point. By Art. 83 a circle can be described to cut three given circles orthogonally. If a point be substituted for one of the circles the construction enables a circle to be described through a given point to cut two given circles orthogonally. Describe such a circle $X$ through $O$ to cut two—and therefore all—circles of the coaxal system orthogonally. Let $P$ be the diametrically opposite point of $O$ on the circle $X$.

Then $O$ and $P$ are conjugate points with respect to all circles of the system (Art. 82). Therefore the polars of $O$ with respect to all circles of the system pass through $P$, which is therefore their envelope.

(*b*) *The locus of the pole of a fixed line with respect to a system of coaxal circles with real limiting points is a conic through these points.*

Let $E$ and $F$ be the limiting points of the coaxal system and let the fixed line meet the perpendiculars through $E$ and $F$ to the line of centres in $K$ and $L$. Then $EK$ and $FL$ are the polars of $F$ and $E$ with respect to all the circles of the system.



The polar of $K$ with respect to any circle $X$ of the system, whose centre is $C$, is a line through $F$ perpendicular to $CK$.

The polar of $L$ with respect to the circle $X$ is a line through $E$ perpendicular to $CL$. The point of intersection $P$ of these lines is the pole of $KL$ with respect to the circle $X$.

For different positions of $C$ on the line of centres the pencils $KC$ and $LC$ are projective. But the pencils $EP$ and $FP$ are projective with these pencils, and therefore are themselves projective. Therefore it will be seen on reference to the definition of a conic section (Art. 72) that the locus of $P$ is a conic through $E$ and $F$.

(*c*) *To find the locus of the points whose polars with respect to three given circles are concurrent,* or

*To find the locus of the common conjugates of three given circles.*

The common conjugate of a point $P$ with respect to two given circles is the diametrically opposite point of the circle through $P$ orthogonal to the two given circles. If this is the conjugate of $P$ with respect to a third circle, the orthogonal circle must likewise be orthogonal to the third circle. Hence the locus of $P$ for common conjugates of three circles is the common orthogonal circle of the three circles. This circle has its centre at the point of intersection of the three radical axes and radius equal to the tangents from this point to the circles. As every circle determines on the line at infinity the same involution, every point on the line at infinity has the same conjugates with regard to the three circles and is therefore part of the locus.

## EXAMPLES.

### General.

(1) To determine the position of the points the ratios of whose distances from three fixed points are given.

To find a point $P$ such that

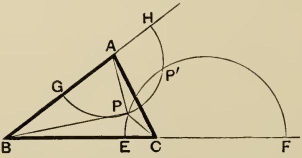
$$\frac{CP}{BP}=k, \quad \frac{AP}{BP}=l,$$

where $ABC$ is any triangle, construct two points $E$ and $F$ on $BC$ such that

$$\frac{BE}{CE}=-\frac{BF}{CF}=+\frac{1}{k}$$

and two points $G$ and $H$ on $BA$ such that

$$\frac{BG}{AG}=-\frac{BH}{AH}=+\frac{1}{l}.$$



Describe circles on $EF$ and on $GH$ as diameters to intersect in $P$ and $P'$. Then $P$ and $P'$ are the required points. (Art. 82.)

(2) To find a point at which three collinear segments subtend equal angles.

Let the segments be $AB$, $A'B'$, $A''B''$. Take $E$ and $F$ common harmonic conjugates of $AA'$, $BB'$, and $G$ and $H$ common harmonic conjugates of $AA'''$ and $BB'''$. Then the circles described on $EF$ and $GH$ as diameters meet at the two required points.

(3) To find the points at which corresponding segments of two projective ranges, determined by points $A$, $A'$, $A''$ and $B$, $B'$, $B''$, whose self-corresponding points are imaginary, subtend equal angles. (Cf. Art. 43.)

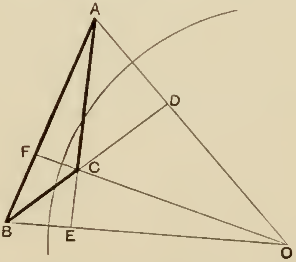
The points obtained in the last example give the solution of the problem.

(4) If $P$, $A$, $B$, $C$, $D$ are five points on a straight line and a circle, which touches the line at $P$, meets again the circles described on $PA$, $PB$, $PC$, $PD$, as diameters in $A'B'C'D'$, then $P(A'B'C'D')=(ABCD)$.

Draw through $P$ a diameter of the circle which touches the line at $P$ to meet it at $Q$. Join $PA'$. Through $A'$ draw a line perpendicular to $PA'$. This line passes through $A$ and $Q$. Similarly $BB'$, $CC'$, $DD'$ pass through $Q$.

Hence $\qquad (ABCD)=(Q\,.\,ABCD)=(Q\,.\,A'B'C'D')=(P\,.\,A'B'C'D')$.

(5) Prove that a circle can be found with respect to which a given triangle is self-polar, and that the circle is real when the triangle is obtuse angled.



Let $D$, $E$, $F$ be the feet of the perpendiculars from the vertices on the opposite sides, and $O$ the orthocentre of the triangle $ABC$. Then

$$OE\,.\,OB=OC\,.\,OF=OD\,.\,OA=K^2$$

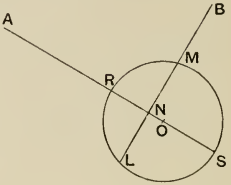
suppose. Hence $BC$, $CA$, $AB$ are the polars of $A$, $B$, $C$ with respect to a circle centre $O$ and radius $K$. This is the required circle.

If the triangle is acute angled $OE\,.\,OB$ is negative and the circle is imaginary but its centre, the orthocentre of the triangle, is real. There is only one circle with regard to which a given triangle is self-conjugate.

(6) The pedal triangle of $ABC$ is $DEF$. $ABC$ is self-polar with regard to a circle $X$. Prove that the polars of $DEF$ with respect to the circle $X$ form a triangle similar to $ABC$ and of twice its dimensions.

$F$ and $C$ are conjugate points with regard to the circle $X$, therefore the polar of $F$ is a line through $C$ perpendicular to $FC$, that is, parallel to $AB$. Hence the polar triangle of $DEF$ is the triangle formed by drawing through $ABC$ lines parallel to opposite sides.

(7) Show that the square of the distance between a pair of conjugate points with respect to a circle is equal to the sum of the powers of the points.

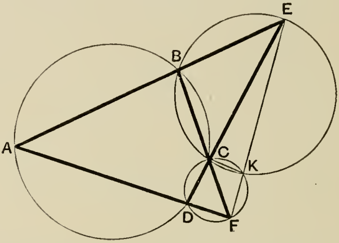


Let the polar of $A$ meet the circle in $L$ and $M$. Take $B$ any point on $LM$. Join $A$ to $O$ the centre of circle $O$ to meet $LM$ in $N$. Then

$$\begin{aligned} AB^2 &= AN^2 + BN^2 \\ &= AN^2 + NM^2 + BN^2 - NM^2 \\ &= AM^2 + BM \cdot BL \\ &= AR \cdot AS + BM \cdot BL. \end{aligned}$$

This theorem may be stated in a different manner, viz.:

If a quadrangle be inscribed in a circle, the square of the distance between two of its diagonal points external to the circle equals the sum of the square of the tangents from these points.



Describe circles round $EBC$ and $FDC$ meeting in $K$. $K$ will be on $EF$ for $CKE + CKF = ABC + ADC =$ two right angles.

Then
$$\begin{aligned} EF^2 &= EK \cdot EF + FE \cdot FK \\ &= EC \cdot ED + FB \cdot FC. \end{aligned}$$

(8) $A$ and $B$ being two given points in the plane of a given circle, find a point $D$ on the circle, such that if $DA$, $DB$ cut the circle again in $E$, $F$, the quadrangle $ABFE$ is inscribable in a circle.

## Coaxal Circles.

(9) If a system of circles be such that the circles of the system have a common orthogonal circle and their centres are collinear, the system is coaxal.

The perpendicular from the centre of the orthogonal circle upon the line of centres is the radical axis.

(10) If the circles of one system cut orthogonally the circles of a second system, both systems are coaxal.

Consider two circles of one system. They determine a radical axis. Since the circles of the other system cut these circles orthogonally, their centres are on this radical axis. Hence their centres are collinear and as they have an orthogonal circle they form a coaxal system. Similarly the other system is coaxal.

(11) To draw through a given point a circle of a non-intersecting coaxal system of circles.

Let $E$ and $F$ be the limiting points of the system. Describe a circle through $P$, the given point, $E$ and $F$, and let the tangent to this circle at $P$ meet the line of centres at $C$. Then the circle whose centre is $C$ and radius $CP$ is the required circle.

(12) To describe a circle of a given coaxal system to touch a given straight line.

Let the given line $p$ meet the radical axis at $P$. Describe a circle orthogonal to the given system with centre $P$ to meet $p$ in $Q$ and $Q'$. If the tangents at $Q$ and $Q'$ to this circle meet the line of centres in $C$ and $C'$, the two circles with centre $C$ and $C''$ and radii $CQ$ and $C'Q'$ comply with the given conditions.

(13) To draw a circle coaxal with two given circles ($X$ and $Y$) to touch another given circle ($Z$).

Describe a circle of the system to meet $Z$ in two real points $P$, $Q$. Let $PQ$ meet the radical axis in $R$. From $R$ draw tangents $RL$, $RM$ to $Z$. Then the circles of the coaxal system through $L$ and $M$ will touch $Z$ at these points.

(14) A variable circle passes through a fixed point $O$ and cuts a given circle in $Q$, $R$, and the tangent at $O$ meets $QR$ in $P$. Show that the locus of $P$ is a straight line.

Since $PQ \,.\, PR = PO^2 =$ the square of the tangent from $P$ to the fixed circle, $P$ lies on the radical axis of $O$ and this circle.

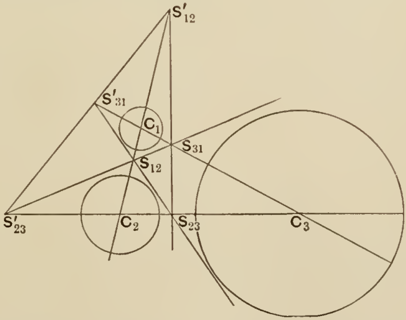
## Centres of Similitude.

(15) If two circles touch two other circles, the chords of contact of either pair and their radical axis pass through a centre of similitude of the other pair.

(16) Find a point where three circles subtend equal angles.

(17) $O$ and $O'$ are the centres of two circles and $T$ is one of their centres of similitude. A straight line $TRSR'S'$ is drawn to meet the circles in $R$, $S$ and $R'$, $S'$, the points being in the order given. Show that the rectangles $TR \,.\, TS'$ and $TS \,.\, TR'$ are equal and constant.

(18) The six centres of similitude of three circles lie three by three on four straight lines termed the axes of similitude.



Let the centres of the three circles be $C_1$, $C_2$, $C_3$, and their radii $r_1$, $r_2$, $r_3$. Let the internal centres of similitude be $S_{12}$, $S_{23}$, $S_{31}$ and the external centres

$S''_{12}$, $S''_{23}$, $S''_{31}$ as in the figure. Then if $C_1C_2C_3$ be taken as triangle of reference, the ratios of $S_{12}$, $S_{23}$, $S_{31}$ are

$$-\frac{r_1}{r_2}, \quad -\frac{r_2}{r_3}, \quad -\frac{r_3}{r_1},$$

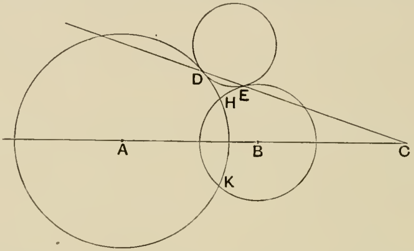
and those of $S'_{12}$, $S''_{23}$, $S''_{31}$ are

$$\frac{r_1}{r_2}, \quad \frac{r_2}{r_3}, \quad \frac{r_3}{r_1}.$$

Hence the product of the ratios of $S'_{12}$, $S''_{23}$, $S''_{31}$ is unity, as likewise are the products of the ratios of

$$S_{12},\ S_{23},\ S''_{31}\ ;\quad S'_{12},\ S_{23},\ S_{31}\ ;\quad S_{12},\ S''_{23},\ S_{31}.$$

(19) Two circles with centres $A$ and $B$ intersect in $H$ and $K$; a third circle touches them at $D$ and $E$ and $DE$ cuts $AB$ in $C$. Show that the circle with centre $C$ which passes through $H$ and $K$ cuts the third circle orthogonally.



The chord of contact $DE$ passes through a centre of similitude of the circles with centres $A$ and $B$ (Example 15) and therefore $CD \,.\, CE = CH^2$. Hence the square of the tangent from $C$ to the third circle equals $CH^2$. Therefore the circle, centre $C$, radius $CH$, is orthogonal to the circle $DE$.

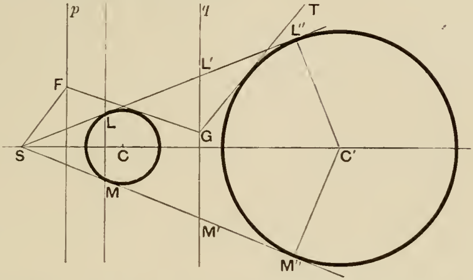
(20) Any line $l$ meets the line joining the centres $C$ and $C'$ of two circles in $L$. $l'$ is the common conjugate of $l$ with regard to the two circles, *i.e.* the connector of $P$ and $P'$ the poles of $l$ with respect to the two circles. $l'$ meets $CC'$ in $L'$. Prove that $L$ and $L'$ are conjugate points of an involution, and that whether or no there are common tangents to the circles, $S$ and $S'$, the centres of similitude of the circles, are the double points of this involution and are the common harmonic conjugates of $C$, $C'$ and $L$, $L'$.

[Cf. Ex. 10, Ch. VII, and (*c*) Art. 106.]

### Circles in Perspective.

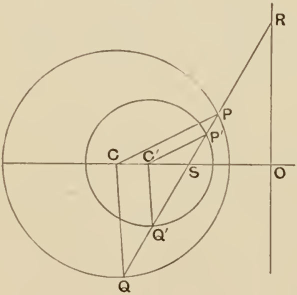
(21) $S$ is a fixed point and $p$, $q$ two given parallel straight lines, $p$ being the radical axis of $S$ and a fixed circle: $FG$ is any tangent to this circle, meeting $p$ in $F$ and $q$ in $G$, and $GT$ is drawn parallel to $SF$. Prove that the envelope of $GT$ is a circle and give a geometrical construction for finding its centre.

Take $S$ as centre of perspective, $q$ as axis and $p$ as vanishing line. Then $GT$ corresponds to $GF$. Hence it envelopes the curve which is the perspective of the circle. But the perspective of a circle is a circle when the vanishing line is the radical axis of the centre of perspective and the circle (Art. 85). Hence $GT$ touches a circle, which touches the tangents from $S$ to the given circle and has for radical axis with the given circle the line $q$.



If $SL$ and $SM$ the tangents from $S$ meet $q$ in $L'$ and $M'$, and $L''$, $M''$ be taken such that $LL'=L''L'$ and $MM'=M''M'$, then $SL$ and $SM$ touch the envelope at $L''$ and $M''$, and its centre $C''$ is the point of intersection of the perpendiculars through $M''$ and $L''$.

(22) To find the centre and axis of perspective of two circles one of which is situated entirely within the other.



Through $C$ and $C'$ the centres of the circles, whose radii are $r$ and $r'$, draw any pair of parallel lines to meet the circles in $P$ and $P'$ and let $PP'$ meet $CC''$ in $S$ and the radical axis in $R$.

Then $\frac{CS}{C'S}=\frac{PS}{P'S}=\frac{CP}{C'P'}=\frac{r}{r'}$. Therefore $S$ is a centre of similitude of the circles and is a fixed point. (Art. 84.)

(1) Take $S$ as centre and the line at infinity as axis of perspective.

Then $(S\infty PP')=\frac{SP}{SP'}=\frac{r}{r'}$. Hence the circles are in perspective.

(2) Let $SP'P$ meet the circles again in $Q$ and $Q'$. Then $\frac{SQ}{SQ'}=\frac{r}{r'}$.

Take $S$ as centre and the radical axis as axis of perspective.

Then
$$\frac{RP}{RQ'}=\frac{RP'}{RQ}=\frac{PP'}{Q'Q}=\frac{SP'-SP}{SQ-SQ'}.$$

But
$$(SRPQ')=\frac{SP}{SQ'}:\frac{RP}{RQ'}=\frac{SP}{SQ'}:\frac{SP'-SP}{SQ-SQ'}=\left(\frac{r}{r'}-1\right):\left(\frac{r'}{r}-1\right)=-\frac{r}{r'}.$$

Therefore the circles are again in perspective.

A second centre of perspective $S'$ may be found by producing $CP$ to meet the larger circle in $P_1$ and determining $S'$ from $P_1$ and $P'$.

# CHAPTER XII

## PROJECTIVE THEOREMS FOR THE CIRCLE :—CARNOT'S THEOREM. PASCAL'S THEOREM. DESARGUES' THEOREM

**89.** The proofs of Carnot's, Pascal's, Desargues' Theorems, and their correlatives for the circle are very similar to those for the conic, but in some cases they take a slightly different form and the proofs are simpler. When proved for the circle they may be deduced for the conic by projection. In this chapter, certain proofs for the circle are given, but the full discussion of these theorems is left over till they are considered for the conic in Chapter XIV.

**Carnot's theorem for the circle and its correlative.**

*If a circle meet the sides $BC$, $CA$, $AB$ of a triangle $ABC$ in three pairs of points,*

$$A_1A_2;\ B_1B_2;\ C_1C_2;$$

*then*

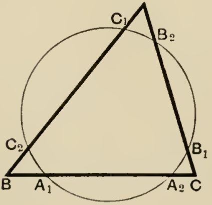
$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2}=1.$$

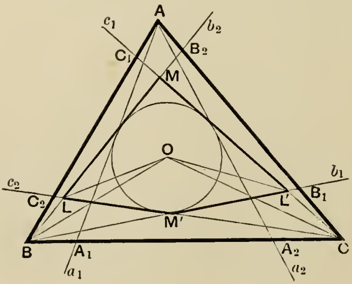
*If the tangents to a circle from the vertices $A$, $B$, $C$ of a triangle $ABC$ meet the opposite sides in three pairs of points,*

$$A_1A_2;\ B_1B_2;\ C_1C_2;$$

*then*

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2}=1.$$





Since the two chords $BC_2C_1$ and $BA_1A_2$ are drawn through the point $B$ to meet the circle in $A_1$, $A_2$ and $B_1$, $B_2$,

$$BA_1 . BA_2 = BC_1 . BC_2.$$

Similarly

$$AC_1 . AC_2 = AB_1 . AB_2,$$

$$CB_1 . CB_2 = CA_1 . CA_2.$$

Therefore

$$\frac{BA_1}{CA_1} \cdot \frac{BA_2}{CA_2} \cdot \frac{CB_1}{AB_1} \cdot \frac{CB_2}{AB_2} \cdot \frac{AC_1}{BC_1} \cdot \frac{AC_2}{BC_2} = 1.$$

Let the points $BB_2 . CC_1$; $BB_2 . CC_2$; $BB_1 . CC_1$; $BB_1 . CC_2$ be denoted by $M$, $L$, $L'$, $M'$, respectively, and let $O$ be the centre of the circle.

By Addendum (11)

$$\frac{BL . BL'}{CL . CL'} = \frac{OB^2}{OC^2}.$$

If the sides of the triangle are denoted by $a$, $b$, $c$ and the tangents by $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, then from consideration of the triangles $BL'C$ and $BLC$

$$\frac{\sin \widehat{ac_1} . \sin \widehat{ac_2}}{\sin \widehat{ab_1} . \sin \widehat{ab_2}} = \frac{BL . BL'}{CL . CL'} = \frac{OB^2}{OC^2}.$$

Similarly

$$\frac{\sin \widehat{ba_1} . \sin \widehat{ba_2}}{\sin \widehat{bc_1} . \sin \widehat{bc_2}} = \frac{OC^2}{OA^2}$$

$$\frac{\sin \widehat{cb_1} . \sin \widehat{cb_2}}{\sin \widehat{ca_1} . \sin \widehat{ca_2}} = \frac{OA^2}{OB^2}.$$

Therefore

$$\frac{\sin \widehat{ac_1}}{\sin \widehat{bc_1}} \cdot \frac{\sin \widehat{ac_2}}{\sin \widehat{bc_2}} \cdot \frac{\sin \widehat{ba_1}}{\sin \widehat{ca_1}} \cdot \frac{\sin \widehat{ba_2}}{\sin \widehat{ca_2}} \cdot \frac{\sin \widehat{cb_1}}{\sin \widehat{ab_1}} \cdot \frac{\sin \widehat{cb_2}}{\sin \widehat{ab_2}} = 1 \quad \text{............(i).}$$

But from the triangles $CC_1A$ and $CC_1B$

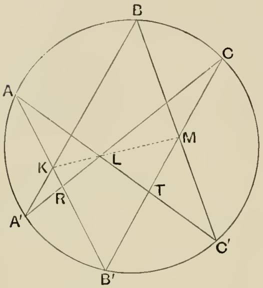
$$\frac{\sin \widehat{bc_1}}{\sin \widehat{ac_1}} = \frac{AC_1}{BC_1} \cdot \frac{\sin A}{\sin B}.$$

Substituting this and the similar values in (i) it follows that

$$\frac{BA_1}{CA_1} \cdot \frac{BA_2}{CA_2} \cdot \frac{CB_1}{AB_1} \cdot \frac{CB_2}{AB_2} \cdot \frac{AC_1}{BC_1} \cdot \frac{AC_2}{BC_2} = 1.$$

## 90. Pascal's Theorem for the circle and its correlative (Brianchon's Theorem).

*If a hexagon be inscribed in a circle the three pairs of opposite sides intersect in three collinear points.*



In the figure let $AC'BA'CB'$ be any inscribed hexagon, and let $K$, $L$, $M$ be the points of intersection of pairs of opposite sides. Let $AB'$ meet $A'C$ in $R$, and $B'C$ meet $AC'$ in $T$.

By the anharmonic property of the circle (Art. 73)

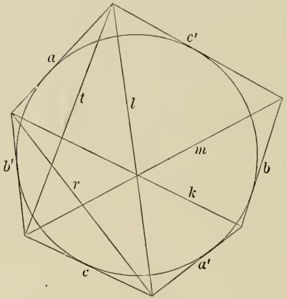
$$(A'.B'CBA) = (C'.B'CBA).$$

Therefore taking intercepts on $AB'$ and $CB'$,

$$(B'RKA) = (B'CMT).$$

Since these ranges have a self-corresponding point at $B'$, the lines $RC$, $KM$, and $TA$, which join corresponding points, are concurrent. Therefore the points $K$, $L$, $M$ are collinear.

*If a hexagon be circumscribed to a circle the connectors of the three pairs of opposite vertices are concurrent.*



In the figure let $ac'ba'cb'$ be any circumscribed hexagon, and let $k$, $l$, $m$ be the connectors of pairs of opposite vertices. Let $ab'.a'c$ be $r$, and $b'c.ac'$ be $t$.

By the anharmonic property of tangents to a circle (Art. 73)

$$(a'.b'cba) = (c'.b'cba).$$

Therefore joining these ranges to $ab'$ and $cb'$,

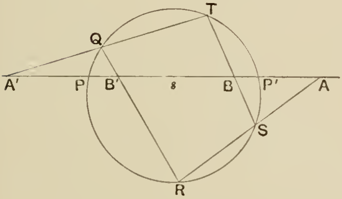
$$(b'rka) = (b'cmt).$$

Since these pencils have a self-corresponding ray in $b'$, the points $rc$, $km$, and $ta$, which are the intersections of corresponding rays, are collinear. Therefore the lines $k$, $l$, $m$ are concurrent.

## 91. Desargues' Theorem for the circle and its correlative.

Desargues' Theorem for a circle takes two different forms:

(i) *If a quadrangle be inscribed in a circle, the pair of points in which this circle is met by any transversal are a pair of conjugate points of the involution determined on that transversal by the sides of the quadrangle.*



Let $Q$, $R$, $S$, $T$ be any four fixed points on a circle and let any transversal $s$ meet the circle in $P$, $P'$ and the lines $QR$, $ST$, $QT$, $RS$ in $B'$, $B$, $A'$, $A$.

Then

$$(Q \,.\, PRP'T) = (S \,.\, PRP'T)$$

by the anharmonic property of the circle.

But $(Q \,.\, PRP'T) = (PB'P'A')$,

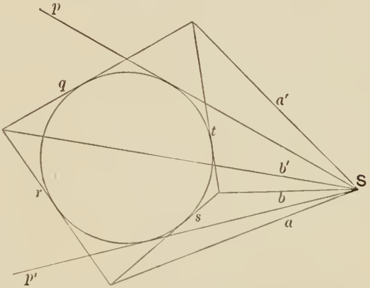
and $(S \,.\, PRP'T) = (PAP'B)$.

Therefore

$$(PB'P'A') = (PAP'B).$$

Therefore $AA'$, $BB'$, $PP'$ are three pairs of conjugate points of an involution.

*If a quadrilateral be circumscribed to a circle, the pair of tangents from any point to the circle are a pair of conjugate lines of the involution determined at that point by the vertices of the quadrilateral.*



Let $q$, $r$, $s$, $t$ be any four fixed tangents to a circle, let the tangents from any point $S$ to the circle be $p$, $p'$ and let the lines joining $S$ to $qr$, $st$, $qt$, $rs$ be $b'$, $b$, $a'$, $a$.

Then

$$(q \,.\, prp't) = (s \,.\, prp't)$$

by the anharmonic property of tangents to a circle.

But $(q \,.\, prp't) = (pb'p'a')$,

and $(s \,.\, prp't) = (pap'b)$.

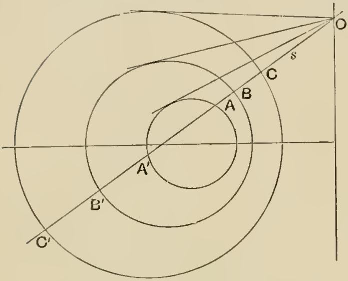
Therefore

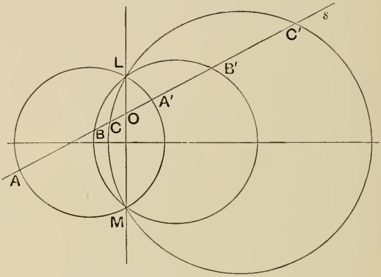
$$(pb'p'a') = (pap'b).$$

Therefore $aa'$, $bb'$, $pp'$ are three pairs of conjugate rays of an involution pencil.

(ii) It will be shown hereafter (Art. 101) that every circle may be regarded as passing through two imaginary fixed points at infinity and that a circle may be considered to be a conic through these two fixed points at infinity. It follows therefore from the general form of Desargues' theorem (Art. 101) that for the circle it may be stated in a second form as follows:

*Every transversal cuts a system of coaxal circles in pairs of conjugate points of an involution.*





Let the transversal $s$ meet the circles in $AA'$, $BB'$, $CC'$, and the radical axis in $O$.

If $O$ is external to the circles the tangents from $O$ to the circles are equal, and the squares of these tangents are equal to $OA \cdot OA'$, $OB \cdot OB'$, $OC \cdot OC'$, .... Hence $OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \ldots$ and the pairs of points $AA'$, $BB'$, $CC'$, ... are pairs of conjugate points of an involution of which $O$ is the centre.

If $O$ is inside the circles, the radical axis must meet the circles in two real points $L$ and $M$. Then

$$OL \cdot OM = OA \cdot OA' = OB \cdot OB' = OC \cdot OC' = \ldots .$$

Hence in this case also $AA'$, $BB'$, $CC'$, ... are pairs of conjugate points of an involution of which $O$ is the centre.

In the former case the double points of the involution are real and in the latter they are imaginary.

## EXAMPLE.

Through $ABC$ the vertices of a triangle parallel lines are drawn cutting the circumcircle in $A'B'C'$. Show that the lines joining $A'$, $B'$, $C'$ to any other point on the circumcircle cut $BC$, $CA$, $AB$, respectively, in points which lie on a straight line parallel to $AA'$.

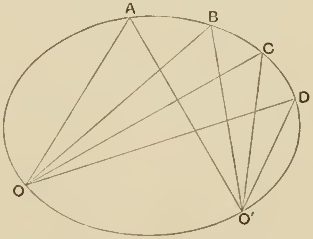
Let $P$ be the point on the circumcircle. Considering the Pascal hexagon $A'P$, $PC'$, $C'C$, $CB$, $BA$, $AA'$, it is seen that two of the points are on a line parallel to $AA'$. By considering the two similar hexagons the result is obtained.

# CHAPTER XIII

## PROJECTIVE FORMS IN RELATION TO THE CONIC:—DEFINITION, CONSTRUCTION AND ANHARMONIC PROPERTY OF THE CONIC. GENERAL PROPERTIES OF THE CONIC. PROPERTIES OF THE FOCI OF A CONIC. ALTERNATIVE DEFINITIONS OF A CONIC

**92. Definition.** A Conic Section, or shortly, a Conic, may be defined in several different ways according to the point of view from which it is considered. *A Conic* is here defined as *the locus of the points of intersection of pairs of corresponding rays of two projective pencils.* In the particular case in which the projective pencils are in perspective the conic becomes a pair of straight lines.

(*a*) *A conic may be described through any five points.*



Let $O$, $O'$, $A$, $B$, $C$ be any five points in a plane. Join $O$ and $O'$ to $A, B, C$. Then $OA, OB, OC$ and $O'A$, $O'B$, $O'C$ determine two projective pencils (Art. 35), the corresponding rays of which intersect in a conic, which passes through $A, B, C$. Since there is a ray of the pencil, whose vertex is $O$, which corresponds to $O'O$ looked upon as a ray of the pencil whose vertex is $O'$, a pair of corresponding rays intersect at $O$. Similarly a pair of corresponding rays intersect at $O'$. Hence the conic passes through $A$, $B$, $C$, $O$ and $O'$.

If three of the five points are collinear—say $A$, $B$, $C$,—the two pencils are in plane perspective. In this case the conic breaks up into the straight line $ABC$ and the line $OO'$ joining the vertices of the pencils.

The conic* obtained as the intersection of the pencils $(O \,.\, ABC\ldots)$ and $(O' \,.\, ABC\ldots)$ is completely determined when $O$, $O'$, $A$, $B$, $C$ are given, for (Art. 35) any number of pairs of corresponding rays may be constructed.

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* There is an exception to this when four of the points $O$, $O'$, $A$, $B$, $C$ are collinear.

(*b*) *The anharmonic ratio of the pencil formed by joining any four fixed points on a conic to a variable point on the conic is constant.*

Let $A, B, C, D$ be any four points of intersection of corresponding rays of the projective pencils, whose vertices are $O$ and $O'$, so that $A$, $B$, $C$, $D$ are any four points on the conic through $OO'ABCD$. Then the anharmonic ratios of the pencils $(O\,.\,ABCD)$ and $(O'\,.\,ABCD)$ are equal.

But the conic is completely determined by the five points $O'$, $A$, $B$, $C$, $D$. Hence $O$ may be looked upon as a variable point on the conic, and the anharmonic ratio of the pencil formed by joining any point $O$ on the conic to any four fixed points $A$, $B$, $C$, $D$ is constant.

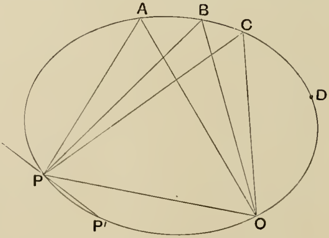
(*c*) *Through five given points, no four of which are collinear, only one conic can be described.*

Let the five points be $A$, $B$, $C$, $O$, $O'$. If two different conics pass through these points, draw through $O'$ a line to meet one conic in $D$ the other in $D'$. Then the anharmonic ratios of the pencils $(O\,.\,ABCD)$ and $(O\,.\,ABCD')$ being equal to that of the pencil $(O'\,.\,ABCD)$ are equal to each other. Hence the points $D$ and $D'$ must coincide.

(*d*) *No straight line can meet a conic in more than two points unless the conic is a pair of straight lines.*

If a line meets a conic in three points $A, B, C$, join $A, B, C$ to any other two points $O$ and $O'$ on the conic. Then the pencils $(O\,.\,ABC\ldots)$ and $(O'\,.\,ABC\ldots)$ are in perspective and the conic becomes the line $ABC$ and the line $OO'$.

(*e*) *There is one and only one tangent at every point on a conic**.



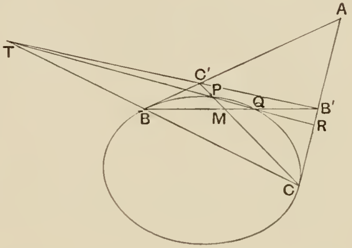
Let $A$, $B$, $C$, $D$, $O$ be the five points which determine the conic, and $P$ any other point on the conic. Then the pencil formed by joining $P$, $A$, $B$, $C$ to any other point $P'$ on the conic has the same anharmonic ratio as $(O\,.\,PABC)$. Let the point $P'$ move indefinitely near to $P$. Then the line $PP'$ becomes the tangent at $P$, and it follows that the tangent at $P$ together with $PA$, $PB$, $PC$ forms a pencil of the same anharmonic ratio as the pencil $(O\,.\,PABC)$.

---

* There is an exception to this if the conic is a pair of straight lines.

Hence, since three rays of this pencil and its anharmonic ratio are given, the fourth ray, which is the tangent at $P$, is uniquely determined.

($f$) *Given two tangents to a conic and their points of contact, to construct the tangent at any point on the conic.*



Let the tangents at $B$ and $C$ to a conic meet at $A$. Take $P$ and $Q$ any two points on the conic and let $CP$, $BQ$ intersect at $M$ and meet $AB$ and $AC$ in $C'$ and $B'$.

Then

$$(B \,.\, BPQC) = (C \,.\, BPQC).$$

Therefore

$$(C'PMC) = (BMQB') = (B'QMB).$$

Hence $C'B'$, $PQ$ and $CB$ are concurrent at some point $T$. Let $P$ and $Q$ coincide. Then $C'B'$, $BC$, and the tangent at $P$ are concurrent. Hence from the harmonic property of the quadrangle, if $TP$ meet $A'C$ in $R$, the range $(ARB'C)$ is harmonic.

Hence to construct the tangent at any point $P$, join $BP$ to meet $AC$ at $B'$ and construct the point $R$ on $AC$, which is the harmonic conjugate of $A$ with respect to $CB'$.

Then $PR$ is the tangent at $P$.

($g$) *Any four fixed tangents to a conic determine a range of constant anharmonic ratio on all other tangents to the conic.*

Let $AC$ and $AB$ be two given tangents to the conic. Take points $P_1, P_2, P_3, P_4$ on the conic. Let $BP_1$, $BP_2$, $BP_3$, $BP_4$ meet $AC$ in $B_1'$, $B_2'$, $B_3'$, $B_4'$ and let the tangents at $P_1$, $P_2$, $P_3$, $P_4$ meet $AC$ in $R_1$, $R_2$, $R_3$, $R_4$. Then the ranges $(AR_1B_1'C)$, $(AR_2B_2'C)$, $(AR_3B_3'C)$, $(AR_4B_4'C)$ are all harmonic. Hence (Art. 14, Ex. (3)) $(R_1R_2R_3R_4) = (B_1'B_2'B_3'B_4')$ = the anharmonic ratio of the pencil formed by joining $P_1$, $P_2$, $P_3$, $P_4$ to any point on the conic. The tangents at $P_1$, $P_2$, $P_3$, $P_4$ therefore determine on any other tangent a range of a constant anharmonic ratio, namely that of the pencil formed by joining $P_1$, $P_2$, $P_3$, $P_4$ to any point on the conic.

**93. Correlative Definition.** The correlative to the definition given in the last Article is as follows. *The envelope of the lines which join pairs of corresponding points of two projective ranges is the correlative of a conic.* It will be shown that the correlative of a conic is a conic, and this will be assumed in the meantime in the statement of certain theorems in this article.

If the ranges are in perspective the conic becomes a pair of points.

In the same way that properties of the conic were deduced from the preceding definition the following may be proved from the correlative definition:

(*a*) *A conic may be described to touch any five lines.*

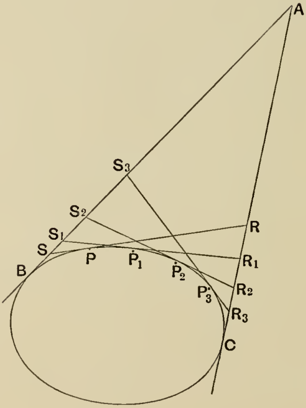
(*b*) *The anharmonic ratio of the range formed by the intersection of any four fixed tangents to a conic with a variable tangent is constant.*

(*c*) *Only one conic can be described to touch five given lines, no three of which are concurrent.*

(*d*) *Through no point can more than two tangents be drawn to a conic.*

(*e*) *Each tangent has one and only one point of contact.*

**The correlative of a conic is a conic.**



Describe a conic through any five points $B$, $C$, $P_1$, $P_2$, $P_3$ and draw the tangents at these points. Let the tangents at $P_1$, $P_2$, $P_3$ meet the tangents $BA$ and $CA$ in $S_1$, $S_2$, $S_3$ and $R_1$, $R_2$, $R_3$, respectively. Describe the correlative of a conic to touch these five lines. Draw $SR$ any other tangent to the conic.

Since

$$(SS_1S_2S_3) = (RR_1R_2R_3)$$

the line $SR$ is a tangent to the correlative of the conic.

Hence any tangent to the conic is also a tangent to the correlative of a conic, which must therefore be a conic.

The anharmonic property of the conic which has already been proved may be summarised as follows:

| | |
|---|---|
| (*a*) **The locus of the points of intersection of corresponding rays of two projective pencils is a conic through the vertices of the pencils.** | (*a*) **The envelope of the lines joining corresponding points of two projective ranges is a conic touching the bases of the ranges.** |

Or conversely:

(*b*) **The anharmonic ratio of the pencil formed by joining any four fixed points on a conic to a variable point on the conic is constant.**

(*b*) **The anharmonic ratio of the range formed by the intersections of any four fixed tangents to a conic with a variable tangent to the conic is constant.**

### 94. Classification of conics.

Conics are divided into different classes according to the nature of their intersections with the line at infinity. There is a point at infinity on a conic determined by each pair of parallel corresponding rays of the generating pencils. On reference to Art. 38 it will be seen that two projective pencils may have two pairs of parallel corresponding rays, one pair of such rays, or no pair.

(i) If the projective pencils have two pairs of parallel corresponding rays the conic is termed *a hyperbola.*

(ii) If they have one pair of such rays the conic is termed *a parabola.*

(iii) If they have no such pair the conic is termed *an ellipse.*

There are two particular cases of the preceding.

(iv) If the projective pencils are oppositely equal the conic is termed *a rectangular hyperbola.*

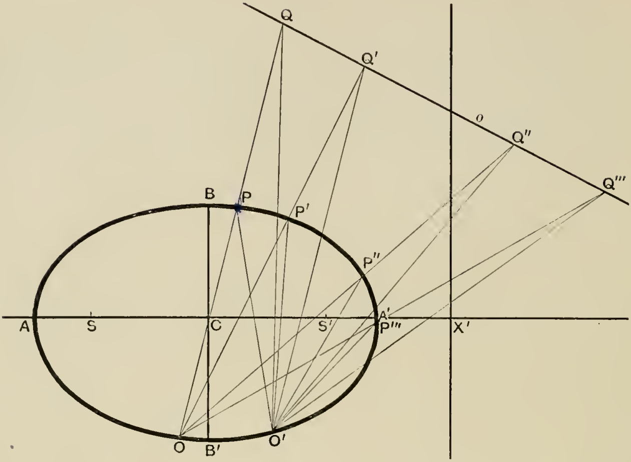
(v) If the projective pencils are directly equal the conic is *a circle.*

In Art. 93 it was shown that the correlative of a conic is a conic. The proof there given does not apply to the case when the conic breaks up into a pair of straight lines. Hence to the above should be added

(vi) If the projective pencils are in plane perspective the conic becomes *a pair of straight lines,* viz. the axis of perspective and the connector of the vertices.

If the ranges, which determine the correlative curve, are in plane perspective, the conic becomes *a pair of points,* viz. the centre of perspective and the point of intersection of the bases.

In Art. 37 it was shown how pencils of this nature could be constructed. The following are the figures of the three kinds of conics constructed by the method there explained as the loci of the points of intersection of projective pencils.


Q
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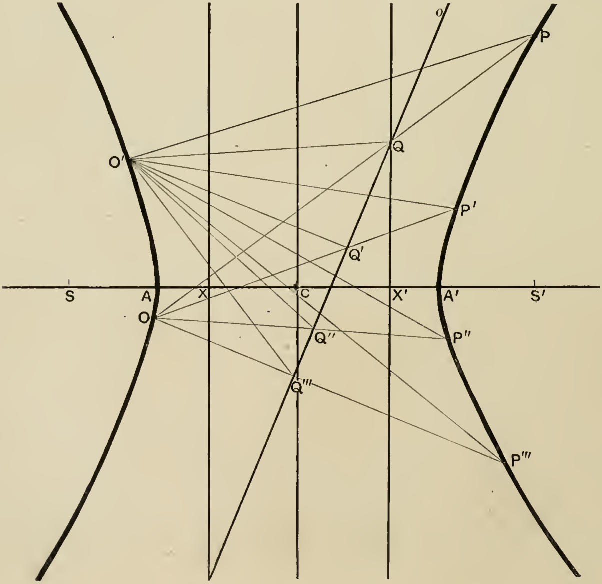
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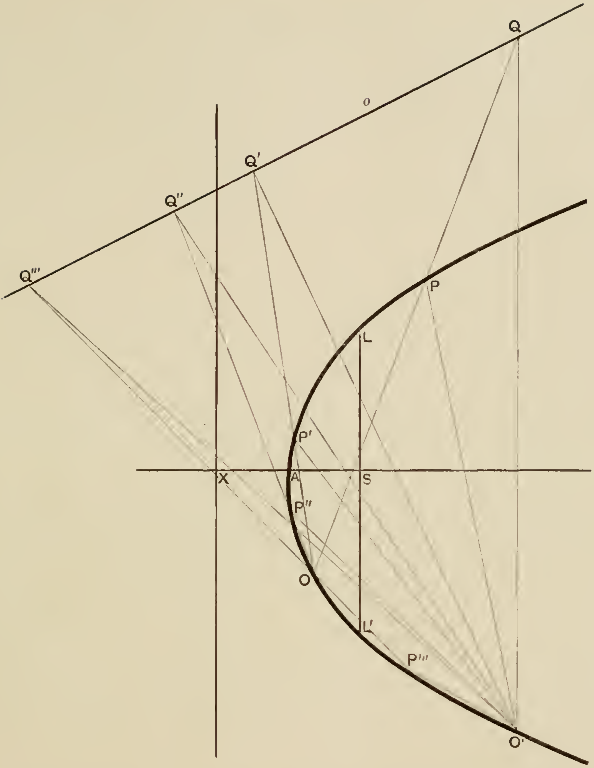

In the figures, which represent respectively an Ellipse, a Hyperbola and a Parabola, $O$ and $O'$ are the vertices of two pencils which have $o$ for axis of perspective. Pairs of corresponding rays intersect in the points $Q$, $Q'$, $Q''$, $Q'''$.... The pencil vertex $O'$ has been rotated round its vertex so that its rays in the new position intersect the corresponding rays of the pencil vertex $O$ in the points $P$, $P'$, $P''$, $P'''$.... These are points on the conics, as also are $O$ and $O'$. In the case of the Ellipse and Hyperbola (see Art. 96), $C$ is the centre, $S$ and $S'$ are the foci, $AA'$ is the major or transverse axis and the perpendiculars through $C$, $X$, and $X'$ are the minor or conjugate axis and the two directrices.

In the Parabola $S$ is the focus, $A$ the vertex, $AS$ the axis and $LSL'$ the latus rectum and the perpendicular through $X$, the directrix.

## 95. Summary of Projective Properties of a Conic.

The projective properties of circles given in Chapter X, which depend on the anharmonic property of the circle, hold for the more general case of the conic. The proofs there given are in such a form as to be applicable in the case of the conic with the mere substitution of the word "conic" for "circle." The student is referred to this chapter for the proofs of the following important properties of the conic.

(*a*) The properties of projective ranges on a conic and of projective systems of tangents. Art. 74.

(*b*) Construction of an involution of points on a conic and of tangents to a conic. Art. 74.

| | |
|---|---|
| (*c*) *If through any point* $S$ *chords be drawn to intersect a conic in* $AA'$, $BB'$, $CC'$, ... *these pairs of points are pairs of conjugate points of an involution on the conic.* | *If from points on a line* $s$ *pairs of tangents* $aa'$, $bb'$, $cc'$, ... *be drawn to a conic these pairs of tangents intersect any other tangent in pairs of conjugate points of an involution.* |
| Conversely:<br>*If* $AA'$, $BB'$, $CC'$, ... *are pairs of conjugate points of an involution on a conic, the lines* $AA'$, $BB'$, $CC'$, ... *are concurrent.* Art. 75. | *If* $aa'$, $bb'$, $cc'$, ... *are pairs of conjugate rays of an involution of tangents to a conic, the points* $aa'$, $bb'$, $cc'$, ... *are collinear.* Art. 75. |
| (*d*) *If through the point of intersection of the tangents at any two points* $T$ *and* $T'$ *on a conic, a chord be drawn to meet the conic in* $A$ *and* $A'$, *then the range* $TT'AA'$ *is harmonic.* Art. 75. | *If from any point on the chord of contact of any two tangents* $t$ *and* $t'$ *to a conic, two other tangents* $a$ *and* $a'$ *be drawn to the conic, then the system of tangents* $tt'aa'$ *is harmonic.* Art. 75. |
| (*e*) *If through any point* $S$ *a variable chord* $SAA'$ *of a conic be drawn the locus of the harmonic conjugate of* $S$ *with regard to* $AA'$ *is a straight line* $s$. Art. 76. | *If from a variable point on a straight line* $s$ *pairs of tangents* $a$, $a'$ *be drawn to a conic the envelope of the harmonic conjugate of* $s$ *with regard to* $aa'$ *is a fixed point* $S$. Art. 76. |
| (*f*) The line $s$ is called the polar of $S$. Art. 76. | The point $S$ is called the pole of $s$. Art. 76. |
| (*g*) Construction for the polar of a given point. Art. 76. | Construction for the pole of a given line. Art. 76. |

($h$) If $P$ and $Q$ are two points such that the polar of one passes through the other, the points are called *conjugate points* with respect to the conic. Art. 76.

If $p$ and $q$ are two straight lines such that the pole of one lies on the other, the lines are called *conjugate lines* with respect to the conic. Art. 76.

($i$) *If $p$ the polar of $P$ passes through $Q$, then $q$ the polar of $Q$ passes through $P$.* Art. 76.

($j$) The construction for pairs of conjugate points, with respect to a conic, on a given straight line; also the converse construction for the pole of a line. Art. 76.

($k$) *A conic determines by means of pairs of collinear conjugate points a definite involution on every straight line in its plane, the double points of which are the points of intersection of the line and conic.* Art. 77.

And the correlative:

*A conic determines by means of pairs of concurrent conjugate lines a definite involution pencil at every point in its plane, the double rays of which are the tangents from the point to the conic.* Art. 77.

($l$) *The anharmonic ratio of a range of four collinear points is equal to that of the pencil formed by their polars.* Art. 77.

($m$) *The anharmonic ratio of four points on a conic is equal to that of the four tangents at these points.* Art. 77.

See also Art. 92 ($g$).

($n$) Properties of pole and polar. Art. 78.

($o$) *Any conic may be looked upon as in harmonic perspective with itself with any point as centre of perspective and the polar of that point as the axis.* Art. 79.

($p$) Properties of complete inscribed quadrangle and complete circumscribed quadrilateral of a conic. Art. 80.

($q$) Construction of a self-polar triangle of a conic. Art. 81.

### 96. Centre, asymptotes, conjugate diameters, axes, and foci of a conic.

**Centre.** It has been proved (Art. 22) that the points at infinity in a plane may be regarded as lying on a straight line termed the line at infinity. Every line in the plane of a conic has a definite point associated with it which is the pole of the line with respect to the conic (Art. 95 ($n$)). The pole of the line at infinity with respect to a conic is termed *its centre.* If $C$ be the centre of a conic and any chord be drawn through it to meet the conic in $P$ and $P'$ and the line at infinity at $\infty$, then $(PP'C\infty)=-1$. Hence $PP'$ is bisected at $C$. Also the tangents at $P$ and $P'$ intersect on the line at infinity and are therefore parallel.

**Asymptotes.** If the generating pencils of a conic have a pair or pairs of parallel corresponding rays, each such pair determines a point of the curve on the line at infinity. In a hyperbola there are two such points. Since the centre is the pole of the line at infinity, the lines through the centre to these points are tangents to the curve, whose points of contact are at infinity. These lines are termed the *asymptotes* of the hyperbola.

In the case of the parabola there is one point at infinity, which must be situated on the line at infinity. A line which meets a conic in only one point must touch the curve. Hence a parabola touches the line at infinity. Since the pole of a tangent is its point of contact the centre of a parabola is the point at infinity on the curve. A straight line which passes through the point at infinity on a parabola is termed a diameter. Such a diameter meets the curve at one point at a finite distance.

The asymptotes of an ellipse are imaginary.

A conic whose centre is at a finite distance, namely an Ellipse or a Hyperbola, is termed a Central Conic.

**Conjugate diameters.** If a diameter $PCP'$ be drawn through the centre $C$ of a central conic, the tangents $p$ and $p'$ at $P$ and $P'$ are parallel and meet at some point, $\infty$, on the line at infinity. The polar of $\infty$ is the diameter $PCP'$. If $\infty C$ meet the curve in $Q$ and $Q'$ then the lines $PCP'$ and $QCQ'$ are termed conjugate diameters and are such that each is parallel to the tangents at the ends of the other. If through $\infty$ a chord $\infty RMR'$ be drawn to meet the conic in $R$ and $R'$ and $PCP'$ in $M$, the range $\infty MRR'$ is harmonic and therefore $M$ bisects $RR'$. But this chord is parallel to $QCQ'$. Hence chords of a conic parallel to a diameter are bisected by its conjugate diameter.

**Axes.** Every pair of conjugate diameters is such that each diameter passes through the pole of the other. Therefore pairs of conjugate diameters are pairs of conjugate rays of an involution of which the centre is the vertex. Such an involution has either one pair of rays at right angles or all pairs of corresponding rays are at right angles (Art. 58). In the general case when there is one pair of conjugate rays at right angles, these are termed the axes of the conic. They are such that the tangents at the points in which each meets the curve are at right angles to the axis in question. If every pair of conjugate rays are at right angles the conic—as can be easily proved—is a circle.

In the case of the ellipse both axes meet the curve in real points and the distances of these points from the centre of the curve are termed the semi-major and semi-minor axes. In the case of the hyperbola one axis, termed the transverse axis, meets the curve in real points. The second axis termed *the conjugate axis* does not meet the curve in real points, but it is convenient to speak of the distance of the centre from the points, where it is met by a parallel to an asymptote drawn through an end of the transverse axis, as a semi-axis of the curve.

In the case of the parabola, if a tangent meet the curve at $P$ and the line at infinity at $\infty$, the polar of $\infty$ is the diameter through $P$. Hence if a chord be drawn through $\infty$ to meet the curve in $R$ and $R'$ and the diameter in $M$, $(\infty MRR')$ is harmonic and $M$ bisects $R$, $R'$. This chord, since it passes through $\infty$, is parallel to the tangent at $P$. Hence chords of a parabola parallel to a tangent are bisected by the diameter through the point of contact of the tangent.

If a line be drawn perpendicular to the diameters of the parabola, it will meet the curve in two points $P$ and $P'$. Let the tangents at $P$ and $P'$ meet at $T$ and through $T$ draw a diameter to meet the curve in $A$. The tangent at $A$ will be parallel to $PP'$. For, since $PP'$ the polar of $T$ passes through the point at infinity on $PP'$, the polar of the point at infinity on $PP'$ is the diameter through $T$ and therefore the tangent at $A$, and the line $PP'$ are parallel. Hence in every parabola there is one point the tangent at which is perpendicular to the direction of the diameters. This point $A$ is called the vertex and the corresponding diameter the axis of the parabola.

**Foci of a conic***. It has been shown Art. 95 ($k$) that a conic determines on every line $p$ in its plane an involution by means of pairs of collinear conjugate points and at every point $P$ in its plane an involution pencil by means of pairs of concurrent conjugate lines. The double rays of this involution pencil, if real, are the tangents from $P$ to the conic. Such an involution pencil has always one pair of conjugate rays at right angles (Art. 58). If the involution pencil has more than one pair of conjugate lines at right angles every pair of conjugate rays is at right angles and every pair of lines at right angles through the vertex $P$ form a pair of conjugate rays of the involution (Art. 58). In this case $P$ the vertex of the involution pencil is termed *a focus of the conic*, and its polar *the corresponding directrix.*

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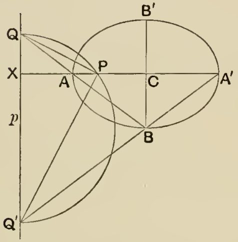
* In Art. 97 a definition of a conic depending on the focus and directrix is given.

A focus must be situated on an axis of the conic, for if the focus be joined to the centre of the conic, the pole of this diameter is at infinity on the perpendicular to the diameter through the focus. Hence the tangents at the ends of the diameter are perpendicular to the diameter, which is therefore an axis. The polar of a focus must also be parallel to an axis, that is, a directrix is parallel to an axis of the conic.

**To find the foci of a central conic.**



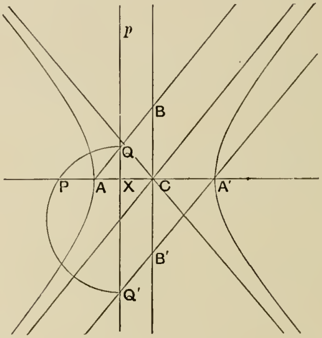
*Ellipse.*

Let $P$ be a focus of an ellipse, whose axes are $AA'$ and $BB'$ and centre $C$. Let the polar $p$ of $P$ meet $AA'$ in $X$. Since the axis $AA'$ passes through $P$ the lines $AB$ and $A'B$ determine a pair of conjugate points $Q$ and $Q'$ on $p$ (Art. 96 ($n$)), which since $P$ is a focus subtend a right angle at $P$. $X$ is the centre of the involution on $p$.

By similar triangles

$$XQ = XA \cdot \frac{CB}{CA}$$

and $$XQ' = XA' \cdot \frac{CB'}{CA'}.$$



*Hyperbola.*

Let $P$ be a focus of a hyperbola, whose transverse axis is $AA'$ and whose centre is $C$. Let the polar $p$ of $P$ meet $AA'$ in $X$. Draw $AQ$ and $A'Q'$ parallel to an asymptote of the hyperbola to meet $p$ in $Q$ and $Q'$ and the perpendicular through $C$ in $B$ and $B'$. Since the axis passes through $P$, $Q$ and $Q'$ are a pair of conjugate points of the involution on $p$, which since $P$ is a focus subtend a right angle at $P$. $X$ is the centre of the involution.

By similar triangles

$$XQ = XA \cdot \frac{CB}{CA}$$

and $$XQ' = XA' \cdot \frac{CB'}{CA'}.$$

$$\therefore\ XQ\,.\,XQ' = -XA\,.\,XA'\,.\,\frac{CB^2}{CA^2};$$

$$\therefore\ XP^2 = (XC^2 - AC^2)\,\frac{CB^2}{CA^2}$$

$$= (XC^2 - CP\,.\,CX)\,\frac{CB^2}{CA^2},$$

since $(XPAA')$ is harmonic.

$$\therefore\ (CX - CP)^2 = CX\,(CX - CP)\,\frac{CB^2}{CA^2},$$

$$\therefore\ CX - CP = CX\,\frac{CB^2}{CA^2} = \frac{CB^2}{CP},$$

$$\therefore\ CX\,.\,CP - CB^2 = CP^2,$$

$$\therefore\ CP^2 = CA^2 - CB^2.$$

$$\therefore\ XQ\,.\,XQ' = XA\,.\,XA'\,.\,\frac{CB^2}{CA^2};$$

$$\therefore\ XP^2 = (CA^2 - CX^2)\,\frac{CB^2}{CA^2}$$

$$= (CX\,.\,CP - CX^2)\,\frac{CB^2}{CA^2},$$

since $(XPAA')$ is harmonic.

$$\therefore\ (CP - CX)^2 = CX\,(CP - CX)\,\frac{CB^2}{CA^2},$$

$$\therefore\ CP - CX = CX\,\frac{CB^2}{CA^2} = \frac{CB^2}{CP},$$

$$\therefore\ CP^2 - CX\,.\,CP = CB^2,$$

$$\therefore\ CP^2 = CA^2 + CB^2.$$

Hence for the ellipse, if $CA > CB$, there are two real foci at equal distances from $C$, and the foci are therefore situated on the major axis. For the hyperbola there are two foci on the transverse axis, that is on the axis which meets the curve in real points.

### To find the focus of a parabola.

Let $P$ be a focus of the parabola and $p$ its polar, then $P$ must be on the axis and $p$ perpendicular to it. Let the axis meet the curve in $A$ and $p$ in $X$. Draw $QPQ'$ perpendicular to the axis and $QT$ and $Q'T'$ parallel to the axis to meet $p$ in $T$ and $T'$.

Then, since $(Q'QP\infty) = -1$,

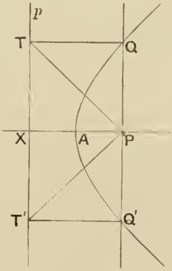
$$PQ = PQ',$$

and since $(XPA\infty) = -1$,

$$XA = AP.$$

Join $TP$ and $T'P$. Then $T$ and $T'$ are conjugate points and, since $P$ is a focus, $TP$ and $T'P$ are at right angles. Therefore $QPXT$ and $Q'PXT'$ are squares and

$$QP = XP = 2\,.\,AP.$$



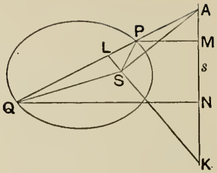
Hence the focus of a parabola is at such a distance from the vertex that the ordinate at the point is double this distance.

### To find the focus of a circle.

From the construction for the foci of an ellipse, it follows that the two foci of a circle coincide at the centre. This also follows from the fact that conjugate lines through the centre of a circle are at right angles.

### Principal properties of the foci of a conic.

(*a*) *If $S$ be a focus, $s$ the corresponding directrix, $P$ any point on the conic, and $PM$ the perpendicular from $P$ on the directrix, then the ratio $\frac{SP}{PM}$ is constant.*



Let the chord $PQ$, joining any two points on the conic, meet $s$ in $A$. Join $SA$, $SP$, $SQ$. Draw $LSK$ perpendicular to $SA$ to meet $PQ$ and $s$ in $L$ and $K$ respectively. Then $SA$ and $SL$, since they are at right angles, are conjugate lines through $S$.

Therefore $ASK$ is a self-conjugate triangle. Hence $SL$ is the polar of $A$ and $(ALPQ)$ is harmonic. Therefore, since $SL$ and $SA$ are at right angles, they are the bisectors of the angle $PSQ$ (Art. 16).

Therefore $\dfrac{SQ}{SP}=\dfrac{QA}{PA}=\dfrac{QN}{PM}$, where $PM$ and $QN$ are perpendiculars on $s$. Hence $\dfrac{SP}{PM}=\dfrac{SQ}{QN}=\text{constant}=e$ (suppose).

(*b*) *In an ellipse the sum of the distances, and in a hyperbola the differences of the distances, of any point on the curve from the foci is constant.*

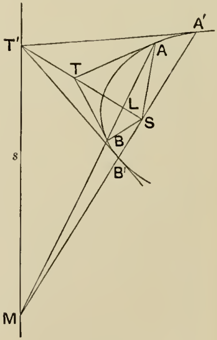
Let $S$ and $S'$ be the foci and $PM$ and $PM'$ the perpendiculars from $P$ on the directrices. It follows for the ellipse that $SP+SP'=e(PM+PM')=e\,.\,MM'$ and for the hyperbola that $SP-SP'=e(PM-PM')=e\,.\,MM'$, where $MM'$ is the perpendicular distance between the directrices.

(*c*) *The tangents from a focus to a conic pass through the circular points at infinity.*

The tangents from a focus are the double rays of the involution of conjugate lines through the focus. Since these conjugate lines are in the case of a focus at right angles, the double rays are the connectors of the focus to the circular points at infinity. (Art. 87.)

(*d*) *A pair of tangents to a conic from any point subtend equal angles at a focus.*

Let $TA$, $TB$ be any pair of tangents to the conic. Let $S$ be a focus, and $s$ its polar. Let $ST$ meet $s$ in $T'$. Let the chord of contact, $A'B'$, of the tangents from $T'$ meet $s$ in $M$. Then $STT'$ is at right angles to $MB'SA'$.



Since the triangle $SMT'$ is self-conjugate, $AB$ passes through $M$, and, if $ST$ meet $AB$ in $L$, $(MLBA)$ is harmonic.

Hence $SL$, $SM$ being at right angles are the bisectors of $ASB$.

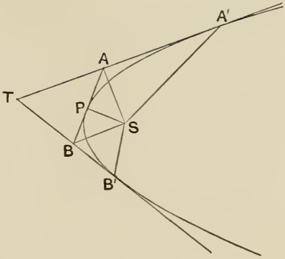
Therefore the angle $BST=$ the angle $AST$.

(*e*) *If a variable tangent meet two fixed tangents in $A$ and $B$ and $S$ be a focus of the conic, then the angle $ASB$ is constant.*

Let $P$ be the point of contact of the variable tangent and $A'$ and $B'$ the points of contact of the fixed tangents $TA$ and $TB$.

Then $$PSB=\tfrac{1}{2}\,.\,B'SP,\ \text{and}\ PSA=\tfrac{1}{2}\,.\,PSA'.$$
$$\therefore\ BSA=\tfrac{1}{2}\,.\,B'SA'.$$

Hence $BSA$ is constant.



Since the angle $ASB$ is constant, the ranges described by $A$ and $B$ on $TA'$ and $TB'$ are projective. Hence, if $AA_1A_2A_3$ and $BB_1B_2B_3$ are two projective ranges on $TA'$ and $TB'$, and a conic be described to touch the lines $TA'$, $TB'$, $AB$, $A_1B_1$, and $A_2B_2$, the line $A_3B_3$ will also touch this conic and, at either of its foci $S$ and $S'$, the segments $AA_1$ and $BB_1$; $A_1A_2$ and $B_1B_2$; $A_2A_3$ and $B_2B_3$; ... will subtend equal angles.

Hence given two projective ranges on different bases two points can be found at which (1) corresponding segments subtend equal angles and (2) the connectors of pairs of corresponding points subtend a constant angle. (See Art. 39 (*a*).)

*Summary.* From the preceding, it is seen that a focus of a conic is:

(1) A point through which pairs of conjugate lines are at right angles.

(2) A fixed point such that the ratio of its distance from any point on the curve to the perpendicular distance of this point from the polar of the fixed point is constant.

(3) A point of intersection of tangents from the circular points at infinity to the conic.

(4) A point at which the intercepts by a variable tangent on any two fixed tangents subtend a constant angle.

### 97. Alternative Definitions of the Conic and the deduction of the Anharmonic Property.

(A) **Definition.** *A conic is the curve obtained from a circle by conical projection or by plane perspective.* (See Art. 19.) As the processes of Projection and Plane Perspective are always reversible, it follows that every conic can be projected into, or its plane perspective formed as, a circle.

The anharmonic property has already been proved for the circle (Art. 73). To prove it for any conic it is only necessary to form the corresponding figure by projection or plane perspective in such a way that the circle is replaced by a conic. Then, since the theorems hold for the circle and the values of anharmonic ratios are unaltered by projection or plane perspective, the theorems hold for the conic.

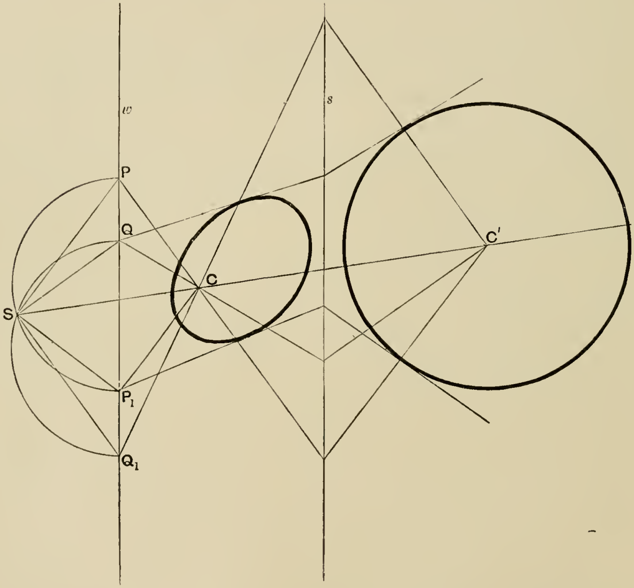
On reference to Art. 73 it will be noticed that the converse theorems are only true in the case of the circle for particular cases of projective ranges and pencils. In fact in order that the locus or envelope may be a circle—which is a particular form of conic—certain special conditions must be fulfilled. In forming the new figure from the one containing the circle the special forms of range and pencil are replaced by the most general forms so that the theorems are true for the conic in the form stated at the end of Art. 92.

*Any conic can be projected into a circle the centre of which is the projection of a given internal point.*

Let $C$ be the point which is to be projected into the centre of the circle and $c$ its polar with respect to the conic. Draw two pairs of conjugate lines through $C$. The angles between these will overlap, since the tangents from this point to the conic are imaginary. Project these angles into right angles and the line $c$ into the line at infinity (Art. 25). The projection of $C$ is the centre of the conic obtained by this projection. The two pairs of conjugate lines through $C$ are projected into two pairs of conjugate diameters of the conic obtained by the projection. Since they are at right angles this conic must be a circle. (Art. 96.)

From the above it is seen that a conic may be projected into a circle and any line, which does not meet the conic in real points, into the line at infinity.

*To form the perspective of a conic so that the corresponding curve may be a circle of which the centre is the point corresponding to a given internal point.*



Let $C$ be the given point which is to correspond to the centre of the circle in the perspective figure. Take its polar as vanishing line $w$. Let the polars of $P$ and $Q$ any two points on $w$ meet $w$ in $P_1$ and $Q_1$. Then $CP$, $CP_1$ and $CQ$, $CQ_1$ are pairs of conjugate lines through $C$. On $PP_1$ and $QQ_1$ describe semicircles intersecting in $S$. Take $S$ as centre of perspective and any line parallel to $w$ as the axis $s$.

In the corresponding figure the point $C'$, which corresponds to $C$, is the centre of the conic and the lines corresponding to $CP$, $CP_1$ and to $CQ$, $CQ_1$ are pairs of conjugate lines through $C'$. They are therefore pairs of conjugate diameters and, since they are at right angles, the conic is a circle.

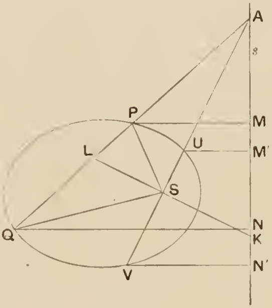
(B) **Definition.** *A conic is the locus of a point which moves so that the ratio of its distance from a fixed point, termed the focus, to its distance from a fixed line, termed the directrix, is constant.*

This ratio is termed the *eccentricity* of the conic. It is less than unity for the Ellipse, greater than unity for the Hyperbola and equal to unity for the Parabola.

It will be proved in the first place that a focus and directrix defined as above comply with the definition of Art. 96.

(*a*) *A point $S$ and a line $s$, such that the distance of $S$ from any point on a conic is in a constant ratio to the distance of the same point from $s$, are a focus and directrix of the conic.*

**Lemma.** If any chord $PQ$ of the conic meet $s$ in $A$, then $SA$ is the external bisector of the angle $PSQ$.



Let $PM$ and $QN$ be the perpendiculars from $P$ and $Q$ on the directrix. Then

$$\frac{SQ}{SP}=\frac{QN}{PM}=\frac{QA}{PA}.$$

Hence, by Euclid VI A, $SA$ is the external bisector of $PSQ$.

Let $SA$ meet the conic in $U$ and $V$ and let $UM'$ and $VN'$ be perpendiculars on the directrix. Then

$$\frac{SV}{SU}=\frac{VN'}{UM'}=\frac{AV}{AU};$$

$$\therefore\ (ASUV)=-1,$$

and, since $AS$ is any line through $S$, $S$ and $s$, the corresponding directrix, are pole and polar.

Draw a line through $S$ perpendicular to $SA$ to meet $AQ$ at $L$.

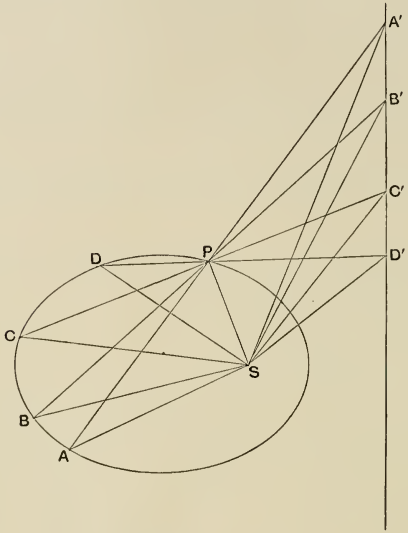
Then $SL$ and $SA$ by the lemma are the internal and external bisectors of $PSQ$, and the range $(ALPQ)$ is harmonic. Also $(ASUV)$ is harmonic. Therefore $SL$ is the polar of $A$. Hence $SL$ and $SA$ are conjugate lines and they are at right angles. Similarly any other pair of conjugate lines through $S$ may be proved to be at right angles. Therefore $S$ is a focus and $s$ its polar is the corresponding directrix.

(*b*) To prove the anharmonic property of a conic directly from the focal definition.

*The anharmonic ratio of the pencil formed by joining any four fixed points on a conic to a variable point on the conic is constant.*

Let $S$ be a focus of the conic and $A$, $B$, $C$, $D$ four fixed points on it. Join $A$, $B$, $C$, $D$ to any point $P$ and let the joining lines meet the directrix in $A'$, $B'$, $C'$, $D'$. Join $S$ to $A$, $B$, $C$, $D$, $A'$, $B'$, $C'$, $D'$.

Then, by the lemma to (*a*), $SA'$, $SB'$, $SC'$, $SD'$ are the external bisectors of the angles $ASP$, $BSP$, $CSP$, $DSP$. Since the angles $A'SB'$, $B'SC'$, $C'SD'$ are half the angles $ASB$, $BSC$, $CSD$ respectively, and these angles are constant, $(S.A'B'C'D')$ is constant for different positions of $P$.



Therefore
$$(P.ABCD)=(P.A'B'C'D')$$
$$=(S.A'B'C'D')$$
$$=\text{a constant.}$$

**98.** *Analytical treatment of the inscribed quadrangle and circumscribed quadrilateral of a conic.*

In the projection of the figure of Art. 80, it should be noticed, that if the triangle $STU$ be taken as triangle of reference, and the ratios of the points in which one of the tangents—say the tangent at $A$—meets its sides be denoted by $a$, $b$, $c$, then the ratios of the points in which the other tangents meet the sides of the triangle can be written down at once.

| | | | |
|---|---|---|---|
| Those for the tangent at $A$ are | $a$, | $b$, | $c$, |
| ” ” ” ” $B$ ” | $a$, | $-b$, | $-c$, |
| ” ” ” ” $C$ ” | $-a$, | $-b$, | $c$, |
| ” ” ” ” $D$ ” | $-a$, | $b$, | $-c$, |

where $abc=1$.

Also, if $A$ determines, by its connectors with the vertices of the triangle of reference, ratios $x$, $y$, $z$ on the opposite sides of the triangle of reference, then similar quantities for $B$, $C$, $D$ may be written down.

| | | | |
|---|---|---|---|
| Those for $A$ are | $x$, | $y$, | $z$, |
| " " $B$ " | $x$, | $-y$, | $-z$, |
| " " $C$ " | $-x$, | $-y$, | $z$, |
| " " $D$ " | $-x$, | $y$, | $-z$, |

where $xyz = -1$.

There is the additional condition that each of the points $A$, $B$, $C$, $D$ is situated on one of the tangents at $A$, $B$, $C$ or $D$.

**Involution on a Conic.**

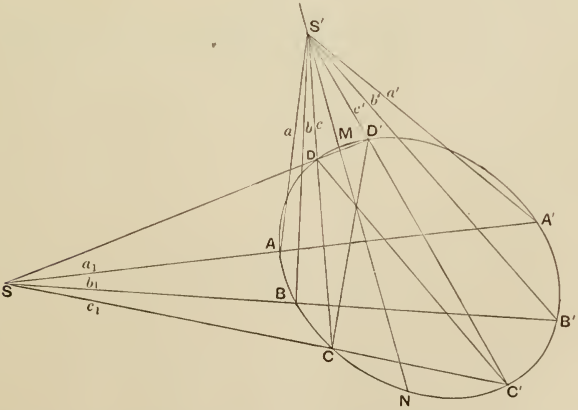
It has been proved (Art. 95 (*c*) and Art. 75) that, if through a point $S$ chords $SAA'$, $SBB'$, $SCC'$ are drawn to meet a conic in the pairs of points $AA'$, $BB'$, $CC'$, these pairs of points are conjugate points of an involution on the conic; and conversely that, if $AA'$, $BB'$, $CC'$ are pairs of conjugate points of an involution on the conic, $AA'$, $BB'$, and $CC'$ are concurrent. The following theorem further illustrates the matter.

(1) *If through any point $S$ chords $a_1$, $b_1$, $c_1$, ... be drawn meeting a conic in $AA'$, $BB'$, $CC'$, ..., and $AA'$, $BB'$, $CC'$, ... be joined to any point $S'$*

(*a*) *on the conic,*

(*b*) *on the polar of $S$,*

*the pencil $aa'$, $bb'$, $cc'$, ... so formed is an involution pencil.*



The first part has already been proved, it is therefore only necessary to prove (*b*). Let $S'$ be any point, not on the conic, such that the pencil $aa'$, $bb'$, $cc'$, ... forms an involution. Let $S'C$ or $c$ meet the conic in $D$, and join $SD$ to meet the conic again at $D'$. Then $D$, $D'$ are a pair of conjugate points of the involution $AA'$, $BB'$, $CC'$, .... Hence $S'C'$ and $S'D'$ are both conjugates of $c$ in the involution $aa'$, $bb'$, .... Therefore $S'$, $D'$, $C'$ are collinear and, from the quadrangle $CC'D'D$, $S'$ must be situated on the polar of $S$, *i.e.* $S$ and $S'$ must be conjugate points with respect to the conic.

When this is the case $S'S$ and $S'MN$, the polar of $S$, are harmonic conjugates of $aa'$, $bb'$, $cc'$, ... and are therefore the double rays of an involution pencil, of which these lines are pairs of conjugate rays.

Conversely *an involution pencil is cut by a conic in pairs of conjugate points* $AA'$, $BB'$, $CC'$, ... *of an involution, when the vertex is situated* (1) *on the conic or* (2) *on the polar of* $AA' . BB'$ *with respect to the conic.*

## EXAMPLES.

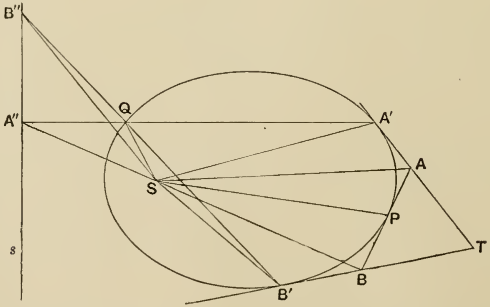
(1) Given six points, explain fully how with ruler and compass to construct the centre and axis of perspective so that the perspectives of the six points shall be one at the centre and the other five on the circumference of a circle.

(2) Tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.

(3) The circumcircle of a triangle circumscribed to a parabola passes through the focus.

(4) Prove that if any point $P$ on a conic be joined to two points $B$ and $C$ on the conic by lines, which meet the tangents at $B$ and $C$ in $C'$, $B'$, and $C'B'$ meets $CB$ in $T$, then $TP$ is the tangent at $P$. (Art. 92.)

(5) If the tangents at any two points $A'$, $B'$ on a conic meet the tangent at any point $P$ in $A$ and $B$, and the lines joining $A'$ and $B'$ to any point $Q$ on the curve meet the directrix corresponding to a focus $S$ in $A''$, $B''$, then the angles $ASB$ and $A''SB''$ are equal.



In the figure $\left.\begin{array}{l} ASP = \frac{1}{2} . A'SP \\ BSP = \frac{1}{2} . B'SP \end{array}\right\}.$

Therefore $ASB = \frac{1}{2} . A'SB'$.

But $SA''$ is the external bisector of $QSA'$

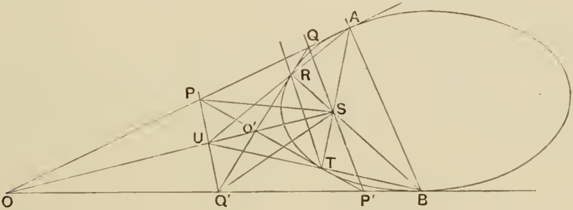
and $SB''$ „ „ „ „ $QSB'$.

Therefore $A''SB'' = \frac{1}{2} . A'SB' = ASB$.

(6) Prove from Example 5 that the anharmonic ratio of four points on a conic is equal to that of the four tangents at the points.

(7) If $CA$ and $CB$ be the semi-transverse and conjugate axes of a hyperbola and points $E$ and $F$ be taken on the transverse axis at distances $\sqrt{CA^2 - CB^2}$ from the centre, prove that the pairs of tangents from these points to the curve are at right angles.

(8) A point at which pairs of corresponding points of two projective ranges on fixed bases subtend a constant angle is a focus of the conic enveloped by the connectors of pairs of corresponding points of the two ranges.



Let $OA$, $OB$ be the given bases, which touch the envelope at $A$ and $B$. Let $S$ be the point at which connectors of pairs of corresponding points subtend a constant angle. Let $AS$ and $BS$ meet the envelope at $R$ and $T$ and let the tangents at these points meet the bases in $Q$, $Q'$ and $P$, $P'$. Then the angles $ASO$, $QSQ'$, $PSP'$, $OSB$ are all equal to some angle $\alpha$.

Let $RQ' . PT$ be $O'$ and $PQ' . OO'$ be $U$, then by the property of inscribed quadrangle and circumscribed quadrilateral

$PQ'$, $AR$, $OO'$ and $BT$ are concurrent at $U$,

$AT$, $QP'$, $RB$ and $OO'$ „ „ „ $S$.

Then the angle $Q'SR$

$$= \alpha - RSQ = USB - BSP' = USP' = \alpha - USP = USA - USP = PSA.$$

Therefore, since $Q'SR = USP' = PSA$, the angles between three pairs of conjugate rays through $S$ are equal and therefore (Example 13, Chapter VIII) the involution pencil at $S$ is orthogonal and $S$ is a focus.

# CHAPTER XIV

## PROJECTIVE FORMS IN RELATION TO THE CONIC:—CARNOT'S THEOREM, PASCAL'S THEOREM, DESARGUES' THEOREM

**99.** Carnot's, Pascal's and Desargues' Theorems, together with the anharmonic property, are the fundamental theorems for the Conic. In this chapter the first three of these theorems are deduced from the anharmonic property of the conic, but on account of their importance alternative proofs are given in each case.

### Carnot's Theorem and its Correlative.

*If a conic meet the sides $BC$, $CA$, $AB$ of a triangle in three pairs of points*

$$A_1, A_2;\ B_1, B_2;\ C_1, C_2,$$

*then*

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2}=1.$$

Converse: *If the above relation holds, a conic can be described through the six points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$.*

*If the tangents from the vertices $A$, $B$, $C$ of a triangle meet the opposite sides in three pairs of points $A_1, A_2;\ B_1, B_2;\ C_1, C_2$,*

*then*

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2}=1.$$

Converse: *If the above relation holds, a conic can be described to touch the six lines $AA_1$, $AA_2$, $BB_1$, $BB_2$, $CC_1$, $CC_2$.*

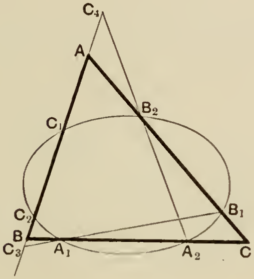
The expression $\dfrac{BA_1}{CA_1}\cdot\dfrac{BA_2}{CA_2}\cdot\dfrac{CB_1}{AB_1}\cdot\dfrac{CB_2}{AB_2}\cdot\dfrac{AC_1}{BC_1}\cdot\dfrac{AC_2}{BC_2}$, where $A, B, C$ are the vertices of any triangle and $A_1, A_2;\ B_1, B_2;\ C_1, C_2$ are any three pairs of points situated on the sides $BC$, $CA$, $AB$ of the triangle respectively, will be termed Carnot's Expression. The value of this expression was shown (Art. 31) to be unaltered by projection. If the ratios of the distance of $A_1$, $A_2$ from the vertices $B$ and $C$ be denoted by $a_1$, $a_2$, the ratios of the distances of $B_1$, $B_2$ from the vertices

$C$ and $A$ by $b_1$, $b_2$, and the ratios of the distances of $C_1$, $C_2$ from the vertices $A$ and $B$ by $c_1$, $c_2$, the expression may be written

$$a_1 a_2 b_1 b_2 c_1 c_2.$$

For the sake of brevity Carnot's Expression will often be quoted in this form, and it should be noticed that, if, for a given triangle $ABC$, the value of Carnot's Expression for the points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ is given, and also the position of five of the points, then the sixth point is uniquely determined.

**Proof from the anharmonic property of a conic.**



In the figure, let $B_1A_1$ meet $AB$ in $C_3$ and $B_2A_2$ meet $AB$ in $C_4$.

Then by the anharmonic property of a conic

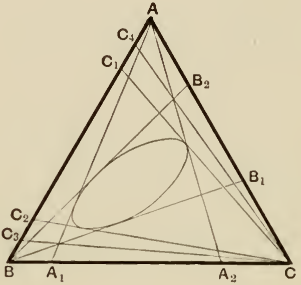
$$(B_1 . C_1C_2B_2A_1) = (A_2 . C_1C_2B_2A_1).$$

Taking intercepts on $AB$

$$(C_1C_2AC_3) = (C_1C_2C_4B).$$

Hence $AB$, $C_1C_2$, $C_3C_4$ are pairs of conjugate points of an involution and therefore (Art. 53)

$$\frac{AC_1}{BC_1} \cdot \frac{AC_2}{BC_2} = \frac{AC_3}{BC_3} \cdot \frac{AC_4}{BC_4} \ldots (1).$$



In the figure, let $AA_1 . BB_1$ and $BB_2 . AA_2$ be projected from $C$ on $AB$ into the points $C_3$ and $C_4$ respectively.

Then by the anharmonic property of tangents to a conic the intersections of $CC_1$, $CC_2$, $AA_1$, $BB_2$ with $AA_2$ and $BB_1$ form two projective ranges.

Projecting these ranges from $C$ on $AB$

$$(C_1C_2AC_4) = (C_1C_2C_3B).$$

Hence $AB$, $C_1C_2$, $C_3C_4$ are pairs of conjugate points of an involution and therefore (Art. 53)

$$\frac{AC_1}{BC_1} \cdot \frac{AC_2}{BC_2} = \frac{AC_3}{BC_3} \cdot \frac{AC_4}{BC_4} \ldots (1).$$

But by Menelaus' Theorem

$$\frac{AC_3}{BC_3} = \frac{1}{\dfrac{BA_1}{CA_1}\cdot\dfrac{CB_1}{AB_1}}$$

and
$$\frac{AC_4}{BC_4} = \frac{1}{\dfrac{BA_2}{CA_2}\cdot\dfrac{CB_2}{AB_2}}.$$

Substituting these values of $\frac{AC_3}{BC_3}$ and $\frac{AC_4}{BC_4}$ in (1),

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2} = 1.$$

But by Ceva's Theorem

$$\frac{AC_3}{BC_3} = \frac{-1}{\dfrac{BA_1}{CA_1}\cdot\dfrac{CB_1}{AB_1}}$$

and
$$\frac{AC_4}{BC_4} = \frac{-1}{\dfrac{BA_2}{CA_2}\cdot\dfrac{CB_2}{AB_2}}.$$

Substituting these values of $\frac{AC_3}{BC_3}$ and $\frac{AC_4}{BC_4}$ in (1),

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}\cdot\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2} = 1.$$

As in the case of the circle, if the sides of the triangle be denoted by $a$, $b$, $c$, and the tangents from the vertices by $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$, the correlative of Carnot's Theorem may be stated in the form

$$\frac{\sin \widehat{ac_1}}{\sin \widehat{ab_1}}\cdot\frac{\sin \widehat{ac_2}}{\sin \widehat{ab_2}}\cdot\frac{\sin \widehat{ba_1}}{\sin \widehat{bc_1}}\cdot\frac{\sin \widehat{ba_2}}{\sin \widehat{bc_2}}\cdot\frac{\sin \widehat{cb_1}}{\sin \widehat{ca_1}}\cdot\frac{\sin \widehat{cb_2}}{\sin \widehat{ca_2}} = 1.$$

*Converse:*

In this case, Carnot's expression for the points $A_1, A_2, B_1, B_2, C_1, C_2$ equals unity.

If a conic be described through the five points $A_1$, $A_2$, $B_1$, $B_2$ and $C_1$, it will meet $C_1C_2$ in a point $C_2'$ such that Carnot's expression for the points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$ and $C_2'$ equals unity.

Hence the points $C_2$ and $C_2'$ must coincide.

In this case, Carnot's expression for the points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ equals unity.

If a conic be described to touch the five lines $AA_1$, $AA_2$, $BB_1$, $BB_2$ and $CC_1$, a second tangent from $C$ to this conic will meet $BC$ in a point $C_2'$ such that Carnot's expression for the points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$ and $C_2'$ equals unity.

Hence the lines $CC_2$ and $CC_2'$ must coincide.

**Proof by Conical Projection or Plane Perspective.**

Let the sides $BC$, $CA$, $AB$ of a triangle meet any conic in $A_1A_2$, $B_1B_2$, $C_1C_2$ respectively.

Let the tangents to any conic from the vertices $A$, $B$, $C$ of a triangle meet the opposite sides in $A_1A_2$, $B_1B_2$, $C_1C_2$ respectively.

Form by conical projection or plane perspective a figure in which the conic is replaced by a circle. (Art. 97.)

The value of Carnot's expression for the new figure will be the same as that for the given figure. (Art. 31.)

But in the new figure, since Carnot's theorem holds for the circle, Carnot's expression is unity. (Art. 89.)

Hence Carnot's theorem holds for the conic.

Form by conical projection or plane perspective a figure in which the conic is replaced by a circle. (Art. 97.)

The value of Carnot's expression for the new figure will be the same as that for the given figure. (Art 31.)

But in the new figure, since the correlative of Carnot's theorem holds for the circle, Carnot's expression is unity. (Art. 89.)

Hence the correlative of Carnot's theorem holds for the conic.

### 100. Pascal's Theorem and its Correlative (Brianchon's Theorem).

*If a hexagon be inscribed in a conic, the three points of intersection of pairs of opposite sides are collinear.*

Converse:

*If two triangles are in perspective, a conic can be described through the six points of intersection of the non-corresponding sides of the triangles.*

*If a hexagon be circumscribed to a conic, the connectors of the three pairs of opposite vertices are concurrent.*

Converse:

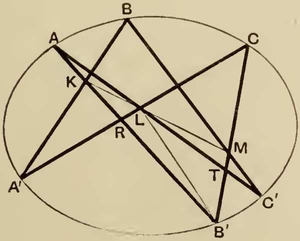
*If two triangles are in perspective, a conic can be described to touch the six lines which join the non-corresponding vertices of the triangles.*

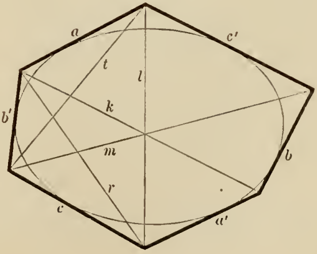
**Proof by the Anharmonic Property of the Conic.**

This proof will be seen to be identical with that given for the circle in Art. 90.

Let $AB'CA'BC'$ be any hexagon inscribed in a conic, and $K$, $L$, $M$ the points of intersection of pairs of opposite sides.

Let $ab'ca'bc'$ be any hexagon circumscribed to a conic, and $k$, $l$, $m$ the connectors of pairs of opposite vertices.





Let $AB'$ meet $A'C$ in $R$ and let $B'C$ meet $AC'$ in $T$. Join $LB'$.

Then

$$(A' \,.\, ABCB') = (C' \,.\, ABCB')$$

by the anharmonic property of the conic.

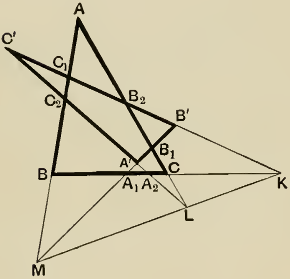
Therefore, taking intercepts on $AB'$ and $CB'$,

$$(AKRB') = (TMCB').$$

Therefore since $B'$ is a self-corresponding point, $AT$, $KM$ and $RC$ are concurrent.

Therefore $K$, $L$, $M$ are collinear.

*Converse :*



Let the corresponding sides of the triangles $ABC$, $A'B'C'$ intersect in the collinear points $K$, $L$, $M$, and the non-corresponding sides in the points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$.

Describe a conic through the five points $C_1$, $B_1$, $B_2$, $A_1$, $A_2$.

Let the line $AB$, which passes through $C_1$, meet the conic again in $C_2'$.

Since $C_1C_2'B_1B_2A_1A_2$ is a hexagon inscribed in a conic, $C_2'A_2$ must pass through $L$.

Let the line joining $ab'$ to $a'c$ be $r$ and that joining $b'c$ to $ac'$ be $t$.

Then

$$(a' \,.\, abcb') = (c' \,.\, abcb')$$

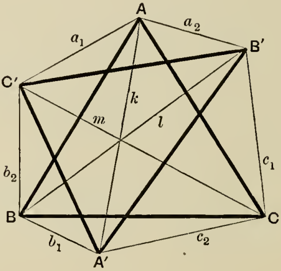
by the anharmonic property of tangents.

Therefore, projecting from $ab'$ and $cb'$,

$$(akrb') = (tmcb').$$

Therefore since $b'$ is a self-corresponding ray, $at$, $km$, and $rc$ are collinear.

Therefore $k$, $l$, $m$ are concurrent.



Let the connectors of the corresponding vertices of the triangles $ABC$, $A'B'C$ be the concurrent lines $k$, $l$, $m$, and those of the non-corresponding vertices $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$.

Describe a conic to touch the five lines $c_1$, $b_1$, $b_2$, $a_1$, $a_2$.

Let the second tangent from $C$ to this conic be $c_2'$.

Since $c_1c_2'b_1b_2a_1a_2$ is a hexagon circumscribed to a conic, $c_2'b_1$ must be on the line $k$.

Therefore $LA_2$ passes through $C_2'$, and $C_2'$ must coincide with $C_2$ the point of intersection of $AB$ and $A_2L$.

Hence the six points $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ are on a conic.

Therefore $c_2'$ must pass through the point $kb_1$. Therefore the tangent $c_2'$ must be the line $c_2$.

Hence the six lines $a_1$, $a_2$, $b_1$, $b_2$, $c_1$, $c_2$ all touch a conic.

### (*a*) **Proof by Conical Projection or Plane Perspective.**

Let $AB'CA'BC'$ be any hexagon inscribed in a conic and $K$, $L$, $M$ the points of intersection of the three pairs of opposite sides.

Form by projection or plane perspective a corresponding figure, wherein the conic is replaced by a circle, and the line $LM$ becomes the line at infinity*. (Art. 97.)

The theorem then depends for its proof on the following, which is proved in the addendum 3 (*a*).

"If two pairs of opposite sides of a hexagon inscribed in a circle are parallel, then the third pair of opposite sides are also parallel."

Let $ab'ca'bc'$ be any hexagon circumscribed to a conic and $k$, $l$, $m$ the connectors of the three pairs of opposite vertices.

Form by projection or plane perspective a corresponding figure, wherein the conic is replaced by a circle, and the point $lm$ becomes the centre of the circle*. (Art. 97.)

The theorem then depends for its proof on the following, which is proved in the addendum 3 (*b*).

"If the lines joining two pairs of opposite vertices of a hexagon circumscribed to a circle pass through the centre, the line joining the third pair of vertices likewise passes through the centre."

### (*b*) **Proof by Carnot's Theorem.**

Let $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$ be the vertices of the hexagon and let the pairs of opposite sides intersect as in the figure in $A'$, $B'$, $C'$. It is required to prove that $A'$, $B'$, $C'$ are collinear.

Consider the triangle $A_1A_2$, $B_1B_2$, $C_1C_2$ whose vertices are $A$, $B$, $C$.

Since $A'$, $B_2$, $C_1$ are collinear, by Menelaus' Theorem

$$\frac{BA'}{CA'} = \frac{1}{\frac{CB_2}{AB_2} \cdot \frac{AC_1}{BC_1}}.$$

Let $A$, $B'$, $C$, $A'$, $B$, $C'$ be the vertices of the hexagon and let $AA'$, $BB'$, $CC'$ meet the sides of the triangle $ABC$ in $A''$, $B''$, $C''$. It is required to prove that $AA'$, $BB'$, $CC'$ are concurrent.

Let the tangents from $A$, $B$, $C$ meet the sides of the triangle $ABC$ in $A_1A_2$, $B_1B_2$, $C_1C_2$ as in the figure.

Since $AA''$, $BB_1$, $CC_2$ are concurrent, by Ceva's Theorem

$$\frac{BA''}{CA''} = -\frac{1}{\frac{CB_1}{AB_1} \cdot \frac{AC_2}{BC_2}}.$$

---

* If the line $LM$ meets the circle in real points in the first case or the point $lm$ in the second case is external to the conic, the construction of the corresponding figure is imaginary and this method does not give a proof of the theorem.

Similarly

$$\frac{AC'}{BC'}=\frac{1}{\dfrac{BA_2}{CA_2}\cdot\dfrac{CB_1}{AB_1}}$$

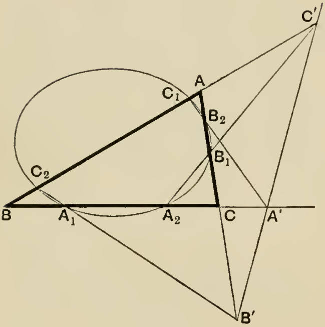
and

$$\frac{CB'}{AB'}=\frac{1}{\dfrac{BA_1}{CA_1}\cdot\dfrac{AC_2}{BC_2}}.$$

Therefore

$$\frac{BA'}{CA'}\cdot\frac{AC'}{BC'}\cdot\frac{CB'}{AB'}$$

$$=\frac{1}{\dfrac{AC_1}{BC_1}\cdot\dfrac{AC_2}{BC_2}\cdot\dfrac{BA_1}{CA_1}\cdot\dfrac{BA_2}{CA_2}\cdot\dfrac{CB_1}{AB_1}\cdot\dfrac{CB_2}{AB_2}}.$$



The expression on the right-hand side is by Carnot's theorem equal to unity. Hence by the converse of Menelaus' theorem $A'$, $B'$, $C'$ are collinear.

Similarly

$$\frac{CB''}{AB''}=-\frac{1}{\dfrac{BA_2}{CA_2}\cdot\dfrac{AC_1}{BC_1}}$$

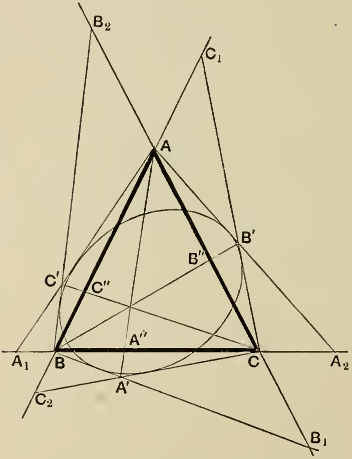
and

$$\frac{AC''}{BC''}=-\frac{1}{\dfrac{CB_2}{AB_2}\cdot\dfrac{BA_1}{CA_1}}.$$

Therefore

$$\frac{BA''}{CA''}\cdot\frac{AC''}{BC''}\cdot\frac{CB''}{AB''}$$

$$=\frac{-1}{\dfrac{AC_1}{BC_1}\cdot\dfrac{AC_2}{BC_2}\cdot\dfrac{BA_1}{CA_1}\cdot\dfrac{BA_2}{CA_2}\cdot\dfrac{CB_1}{AB_1}\cdot\dfrac{CB_2}{AB_2}}.$$



The expression on the right-hand side is by the correlative of Carnot's theorem equal to $-1$. Hence by the converse of Ceva's theorem $AA'$, $BB'$, $CC'$ are concurrent.

The proof of the converse theorems by means of Carnot's theorem is left as an exercise for the student.

(*c*) **Proof by Projective ranges on the Conic.**

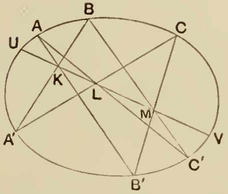
Let the inscribed hexagon be $AB'CA'BC'$ and let $K$, $L$, $M$ be the points $BA'.AB'$; $A'C.AC'$ and $BC'.CB'$.

Let the circumscribed hexagon be $ab'ca'bc'$ and let $k$, $l$, $m$ be the lines $ba'.ab'$; $a'c.ac'$ and $bc'.cb'$. It is

It is required to prove that these points are collinear.

The two groups of three points $A$, $B$, $C$ and $A'$, $B'$, $C'$ determine two projective ranges on the conic. (Art. 95 ($a$).)

Consider the pencils $B.A'B'C'$ and $B'.ABC$. They have a self-corresponding ray in $BB'$ and are therefore in perspective. Hence every pair of corresponding rays of these pencils intersects on $KM$. If $KM$ meets the conic in $U$ and $V$*, $BU$ and $BV$ correspond to $B'U$ and $B'V$. Therefore $U$ and $V$ are self-corresponding points of the projective ranges determined on the conic by the pencils with vertices $B$ and $B'$, that is by the pairs of corresponding points $AA'$, $BB'$, $CC'$.



Hence, when the ranges $ABC$... and $A'B'C'$... are given, $U$ and $V$ are fixed, as is also the line $KM$.

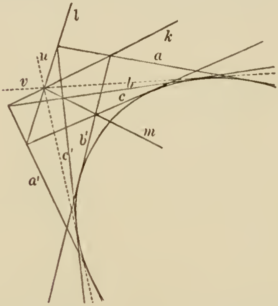
Similarly by considering the pencils $A'.ABC$... and $A.A'B'C'$... it is seen that the point $L\,(AC'.CA')$ is on $UV$. Hence $K$, $L$, $M$ are collinear.

This proof should be compared with that given (Art. 36) for the corresponding theorem in which the vertices of the hexagon lie on a pair of straight lines.

required to prove that these lines are concurrent.

The two groups of three tangents $a$, $b$, $c$ and $a'$, $b'$, $c'$ determine two projective systems of tangents to the conic. (Art. 95 ($a$).)

Consider the ranges $b.a'b'c'$ and $b'.abc$. They have a self-corresponding point $bb'$ and are therefore in perspective. Hence every pair of corresponding points of these ranges is collinear with $km$. If $u$ and $v$* are tangents from $km$ to the conic, $bu$ and $bv$ correspond to $b'u$ and $b'v$. Therefore $u$ and $v$ are the self-corresponding rays of the two projective pencils of tangents to the conic determined by the ranges on $b$ and $b'$ and therefore by the pairs of corresponding tangents $aa'$, $bb'$, $cc'$.



Hence, when the systems of tangents $abc$... and $a'b'c'$... are given, $u$ and $v$ are fixed, as is also the point $km$.

Similarly by considering the ranges $a'.abc$... and $a.a'b'c'$... it is seen that the line $l\,(ac'.ca')$ passes through $uv$. Hence $k$, $l$, $m$ are concurrent.

This proof should be compared with that given (Art. 36) for the corresponding theorem in which the sides of the hexagon pass through two points.

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* This proof only holds when the line $KM$ meets the conic in real points.

* This proof only holds when the tangents from $km$ to the conic are real.

### 101. Desargues' Theorem and Correlative.

*Any transversal is cut by a system of conics, through four fixed points, in pairs of conjugate points of an involution.*

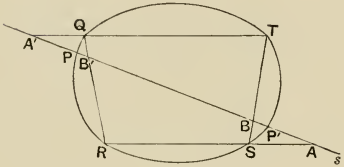
Converse: *Conics described through the vertices of a triangle and through pairs of conjugate points of an involution range intersect in a fourth fixed point.*

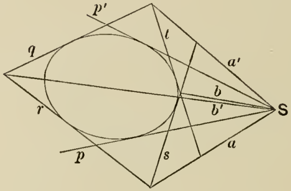
*The pairs of tangents from any point to a system of conics, touching four fixed lines, are pairs of conjugate rays of an involution.*

Converse: *Conics described so as to touch the sides of a triangle and also a pair of conjugate rays of an involution pencil touch a fourth fixed line.*

### Proof by the Anharmonic property of the Conic.

This proof should be compared with that for the circle in Art. 91 (i).





Let $Q$, $R$, $S$, $T$ be any four fixed points and let a conic through them meet any transversal $s$ in $P$ and $P'$. Let $QR$, $ST$, $QT$ and $RS$ meet $s$ in $B'$, $B$, $A'$ and $A$.

Then $(Q . PRP'T) = (S . PRP'T)$.

But $(Q . PRP'T) = (PB'P'A')$

and $(S . PRP'T) = (PAP'B)$.

Therefore

$$(PB'P'A') = (PAP'B).$$

Therefore $AA'$, $BB'$, $PP'$ are pairs of conjugate points of an involution.

If any other conic be described through $Q$, $R$, $S$, $T$, it will meet $s$

Let $q$, $r$, $s$, $t$ be the four fixed lines and let the tangents from any point $S$ to a conic, which touches these lines, be $p$ and $p'$. Let the lines joining $qr$, $st$, $qt$ and $rs$ to $S$ be $b'$, $b$, $a'$ and $a$.

Then $(q . prp't) = (s . prp't)$.

But $(q . prp't) = (pb'p'a')$

and $(s . prp't) = (pap'b)$.

Therefore

$$(pb'p'a') = (pap'b).$$

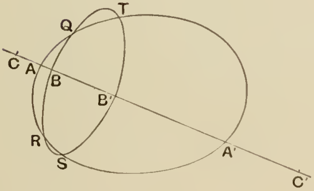
Therefore $aa'$, $bb'$, $pp'$ are pairs of conjugate rays of an involution pencil.

If any other conic be described to touch $q$, $r$, $s$, $t$, the tangents in a pair of points of the same involution, namely that determined by the two pairs of conjugate points $A$, $A'$ and $B$, $B'$.

The points where $SQ$ and $RT$ meet $s$ are also a pair of conjugate points of the same involution. (Art. 56.)

*Converse:*

Let the vertices of the triangle be $Q$, $R$, $S$ and let the involution be determined by the pairs of conjugate points $A$, $A'$ and $B$, $B'$.



Describe two conics through the points $A$, $A'$, $Q$, $R$, $S$ and $B$, $B'$, $Q$, $R$, $S$ respectively to intersect in some point $T$.

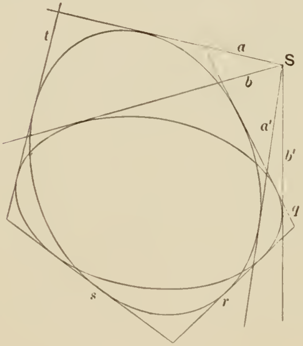
Describe a third conic through $Q$, $R$, $S$, $T$ and $C$, some given point on the transversal.

This conic meets the transversal in $C'$ the conjugate of $C$ in the involution determined by $A$, $A'$ and $B$, $B'$. Since only one conic can be

from $S$ will be a pair of conjugate rays of the same pencil, namely that determined by the two pairs of conjugate rays $a$, $a'$ and $b$, $b'$.

The lines joining $S$ to $sq$ and $rt$ are also a pair of conjugate rays of the same involution. (Art. 56.)

Let the sides of the triangle be $q$, $r$, $s$ and let the involution be determined by the pairs of conjugate lines $a$, $a'$ and $b$, $b'$.



Describe two conics to touch the lines $a$, $a'$, $q$, $r$, $s$ and $b$, $b'$, $q$, $r$, $s$ respectively. Draw the fourth common tangent, $t$, to these conics.

Describe a third conic to touch $q$, $r$, $s$, $t$ and $c$, some given line through $S$, the centre of the involution.

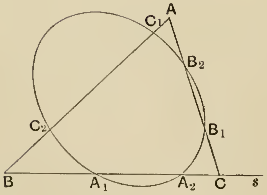
This conic will touch $c'$ the conjugate to $c$ in the involution determined by $a$, $a'$ and $b$, $b'$. Since only one conic can be described through five points, this conic may be looked upon as the conic through $Q$, $R$, $S$, $C$ and $C'$. Since this conic passes through $T$, the theorem is proved.

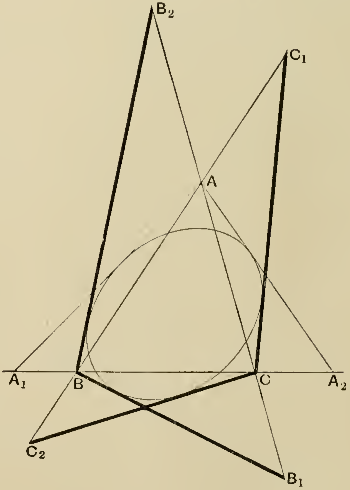
described to touch five lines, this conic may be looked upon as the conic touching $q$, $r$, $s$, $c$ and $c'$. Since this conic touches the line $t$, the theorem is proved.

(*a*) **Proof by Projection or Plane Perspective.**

Desargues' Theorem is related to the theorem, already proved in Art. 91 (ii), that a system of coaxal circles is cut by every transversal in pairs of conjugate points of an involution. The connexion may be shown by an imaginary projection as follows. Project one of the conics through the four points into a circle and the connector of two of the four given points into the line at infinity*. These two points then become the circular points at infinity (Art. 87) and, since the projections of all the conics pass through these points, the conics all become circles (see page 189). The circles all pass through two other points and therefore form a system of coaxal circles. Since these circles are cut by every transversal in pairs of conjugate points of an involution, the conics in the original figure are all cut in a similar manner by every transversal.

(*b*) **Proof by Carnot's Theorem.**





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* This involves an imaginary projection the use of which has not been justified.

Let $C_1$, $C_2$, $B_1$, $B_2$ be the four fixed points through which the conics are described. Let one of the conics meet the transversal $s$ in $A_1$, $A_2$ and another conic (not shown in the figure) meet $s$ in $A_1'$, $A_2'$. Let $B_1B_2$ and $C_1C_2$ intersect at $A$ and meet $s$ in $C$ and $B$.

Then by Carnot's theorem

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}=\frac{1}{\dfrac{CB_1}{AB_1}\cdot\dfrac{CB_2}{AB_2}\cdot\dfrac{AC_1}{BC_1}\cdot\dfrac{AC_2}{BC_2}}=\frac{BA_1'}{CA_1'}\cdot\frac{BA_2'}{CA_2'}.$$

Therefore $(BCA_1A_1')=(CBA_2A_2')$.

Hence $B$, $C$; $A_1$, $A_2$; $A_1'$, $A_2'$ are pairs of conjugate points of an involution. Since $B$, $C$ and $A_1$, $A_2$ may be regarded as fixed, any other conic through $B_1$, $B_2$, $C_1$, $C_2$ determines on $BC$ pairs of points, which are conjugates in the same involution.

Let $BB_2$, $BB_1$, $CC_2$, $CC_1$ be the four fixed lines which are touched by the conics. Let the pair of tangents from any point $A$ to one conic meet $BC$ in $A_1$, $A_2$ and those from $A$ to another conic (not shown in the figure) meet $BC$ in $A_1'$, $A_2'$.

Then by the correlative to Carnot's theorem

$$\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}=\frac{1}{\dfrac{CB_1}{AB_1}\cdot\dfrac{CB_2}{AB_2}\cdot\dfrac{AC_1}{BC_1}\cdot\dfrac{AC_2}{BC_2}}=\frac{BA_1'}{CA_1'}\cdot\frac{BA_2'}{CA_2'}.$$

Therefore $(BCA_1A_1')=(CBA_2A_2')$.

Hence $B$, $C$; $A_1$, $A_2$; $A_1'$, $A_2'$ are pairs of conjugate points of an involution. Since $B$, $C$ and $A_1$, $A_2$ may be regarded as fixed, any other conic touching $BB_1$, $BB_2$, $CC_1$, $CC_2$ determines by tangents from $A$ pairs of lines, which are conjugates in a given involution.

### Carnot's, Pascal's and Desargues' Theorems for a pair of lines and their correlatives for a pair of points.

In Art. 94 it was pointed out that a pair of straight lines could be regarded as a conic and a pair of points as the correlative of a conic. On reference to the earlier chapters it will be seen that the theorems proved in this chapter are true, as might be expected, for a pair of lines, and their correlatives for a pair of points. Thus

Carnot's theorem for a pair of lines follows at once from Menelaus' theorem. (Art. 13 (*d*).)

Pascal's theorem for a pair of lines becomes the theorem proved in Art. 36.

Desargues' theorem for pairs of lines becomes the Involution Property of a quadrangle proved in Art. 56.

Carnot's theorem for a pair of points follows at once from Ceva's theorem. (Art. 13 (*c*).)

Brianchon's theorem for a pair of points becomes the theorem proved in Art. 36.

The correlative of Desargues' theorem for a pair of points becomes the Involution Property of a quadrilateral proved in Art. 56.

The anharmonic property of the conic and also the involution property hold in these particular cases.

### Properties of the Conic and Circle involving the imaginary.

The consideration of the properties of the circle and conic, which depend on the imaginary, has been postponed for later consideration, but in connexion with the solution of problems it is useful to be able to quote certain of the results which may be obtained. A short summary of these is therefore given.

It has been proved (Art. 95 ($k$)) that every conic and every circle determines on each line in its plane a definite involution. Such an involution has two real, two coincident, or two imaginary double points. If the two double points are real they are the points of intersection of the line and conic (Art. 95 ($k$)). If they are coincident the line touches the conic and the pair of coincident double points become the point of contact. If the double points are imaginary, which is the case when the pairs of conjugate points of the involution overlap, the determination of the double points involves $\sqrt{-1}$ and graphically the line does not meet the conic. Such a pair of imaginary points, termed *conjugate imaginary points*, may be regarded, from a certain point of view, as on the curve. Most theorems which are true for a pair of real points on the conic or circle, have an interpretation when a pair of such imaginary conjugate points are substituted for these real points.

The statement that the imaginary (or real) double points of an involution are on a conic is equivalent to saying that the conic in question is such that the pairs of conjugate points of the involution are pairs of conjugate points with regard to the conic. The student is strongly advised to bear in mind this point of view, which removes many difficulties.

Correlatively a conic or circle determines at every point in its plane an involution pencil by means of pairs of conjugate lines passing through a point $P$ (Art. 95 ($k$)). The double rays of this involution pencil, if real, are tangents from $P$ to the conic and, if imaginary, they are the imaginary tangents from $P$. Thus the statement that a conic touches two such imaginary lines may be interpreted as meaning that the pairs of conjugate rays of the involution, of which these lines are double rays, are conjugate lines with respect to the conic.

It has been shown that conics are divided into different classes according to the nature of their intersections with the line at infinity. All circles determine on the line at infinity the same involution, pairs of conjugate points of which are determined by pairs of orthogonal diameters (Art. 87). Hence, with the assumption of the previous paragraph, every circle meets the line at infinity in the same pair of imaginary points, termed the *circular points at infinity*, and every pair of orthogonal lines determines on the line at infinity a pair of points which are harmonic conjugates of these circular points at infinity. Conversely every conic through the circular points at infinity is a circle.

It is usual to assume that any pair of points in a plane may be projected into the circular points at infinity. On reference to Art. 60 it will be seen that any two involutions with real double points may be projected into each other and likewise any pair of involutions with imaginary double points. Hence any pair of conjugate imaginary points may be projected into the circular points at infinity. The projection of an involution with real double points into an involution with imaginary double points is a process most difficult to justify involving as it does the change of the other points from real to imaginary points. Yet it is a fact that in most theorems a pair of real points may be replaced by the circular points at infinity or by any other pair of conjugate imaginary points. Probably the true explanation of this lies, not in the fact that real points may be projected into imaginary points, but in the fact that every conic is met by some lines in real and by some lines in imaginary points, and that the properties of the involutions determined on these lines are similar.

The usual method for projecting two real points $A$ and $B$ into the circular points at infinity is as follows.

"Describe a conic through these points. Project this conic into a circle and the line $AB$ into the line at infinity (Art. 97)." This can only be done by an imaginary projection. "Then the projections of $A$ and $B$ are the circular points at infinity and every conic through these points is a circle."

The circular points at infinity possess another remarkable property. Pairs of conjugate lines through a focus are at right angles. Hence the tangents to a conic from a focus, which are the double rays of the involution pencil formed by these conjugate lines, pass through the circular points at infinity. Therefore the foci of a conic are the points of intersection of tangents to the conic from the circular points at infinity. One pair are real and are the real foci (Art. 96) of the conic. The other pair of points of intersection are imaginary and are a pair of imaginary foci of the conic, which are situated on the minor or conjugate axis.

# CHAPTER XV

## DEDUCTIONS FROM THE ANHARMONIC PROPERTY OF THE CONIC, FROM THE POLE AND POLAR PROPERTIES OF THE CONIC, AND FROM CARNOT'S THEOREM

### 102. Anharmonic Property of the Conic.

### Particular Cases.

In dealing with deductions from the anharmonic property of the conic and its correlative an important special case should be noticed. In an anharmonic ratio, say $(ABCD)$, it is often possible to arrange for one of the elements $D$ to be at infinity, in which case the anharmonic ratio reduces to the simple ratio $\frac{AC}{BC}$.

Similarly in the case of a pencil $abcd$ whose anharmonic ratio is $\frac{\sin \widehat{ac}}{\sin \widehat{bc}} : \frac{\sin \widehat{ad}}{\sin \widehat{bd}}$, if $d$ be the internal bisector of the angle $\widehat{ab}$, $\frac{\sin \widehat{ad}}{\sin \widehat{bd}} = -1$, and if $d$ be the external bisector of the angle $\widehat{ab}$, $\frac{\sin \widehat{ad}}{\sin \widehat{bd}} = +1$. In the former case the anharmonic ratio reduces to $-\frac{\sin \widehat{ac}}{\sin \widehat{bc}}$ and in the latter to $\frac{\sin \widehat{ac}}{\sin \widehat{bc}}$.

### Deductions and Examples.

*Parabola.*

(1) If two fixed points $A$ and $B$ on a parabola be joined to a variable point $P$ on the curve and $PA$ and $PB$ meet the diameter through a fixed point $C$ in $A'$ and $B'$, then $CA' : CB'$ is constant.

If $\infty$ be the point at infinity on the diameter through $C$, then

$(P \, . \, C\infty \, A'B') = \text{constant}$.

Two fixed tangents to a parabola are cut proportionally by tangents to the curve.

If $\infty$ represent the line at infinity, which is a tangent, $p$ and $q$ the two fixed tangents, and $a$, $b$, $c$ any three tangents, then $(p \, . \, abc\infty) = (q \, . \, abc\infty)$.

*Conversely*, the lines joining corresponding points of two similar ranges envelope a parabola.

(2) If a diameter of a parabola meets the curve in $P$, the two tangents at $A$ and $B$ in $A'$ and $B'$ and their chord of contact at $C$, then $PC^2=PA'.PB'$.

Let $\infty$ be the point at infinity on the diameter, then

$$(A\,.\,ABP\infty)=(B\,.\,ABP\infty),$$

and taking intercepts on the diameter

$$(A'CP\infty)=(CB'P\infty),$$

$$\therefore\ \frac{A'P}{CP}=\frac{CP}{B'P},\qquad \therefore\ CP^2=PA'.PB'.$$

*Hyperbola.*

(3) Parallels to the asymptotes through any point on a hyperbola cut any semi-diameter so that it is a mean proportional between the segments on it measured from the centre.

Let $A$ be the end of the diameter, $O$ the centre, $P$ the point on the curve, $P'$ and $P''$ the points where lines through $P$ parallel to the asymptotes meet $OA$, $\infty$ and $\infty'$ the points at infinity on the curve.

Then

$$(\infty\,.\,AP\infty\infty')=(\infty'\,.\,AP\infty\infty'),$$

or taking intercepts on $OA$,

$$\frac{AO}{P'O}=\frac{P''O}{AO}.$$

A variable tangent to a hyperbola meets the asymptotes in two points the product of whose distances from the centre is constant.

Let $a$ and $b$ be any two tangents and $a_1$ and $a_2$ the asymptotes. Then

$$(a_1\,.\,aba_1a_2)=(a_2\,.\,aba_1a_2).$$

Taking intercepts on the asymptotes the required result is obtained.

(4) If through a fixed point $O$ a variable line be drawn to meet two fixed lines in $A$ and $B$, and a point $P$ be taken on this line such that $OP^2=OA.OB$, then the locus of $P$ is a hyperbola, whose asymptotes are parallel to the given lines.

Converse of (3).

If points $A$ and $B$ be taken on two fixed lines such that, if $O$ be the point of intersection of the lines, $OA.OB$ is constant, then the envelope of the line $AB$ is a hyperbola, having $OA$ and $OB$ for asymptotes.

Converse of (3).

(5) If two fixed points $A$ and $B$ on a hyperbola be joined to a variable point $P$ and the joining lines meet a parallel to an asymptote through a point on the curve $C$ in $A'$ and $B'$, then the ratio $CA':CB'$ is constant.

(6) The lines joining two fixed points on a hyperbola to a variable point on the curve intercept a constant length on either asymptote.

In example (5) if $A$ and $B$ be joined to another point $P'$ on the curve so as to meet the parallel to the asymptote in $A''$, $B''$, $\frac{CA'}{CB'}=\frac{CA''}{CB''}=\frac{A'A''}{B'B''}$. If $C$ be taken at infinity, $A'A''=B'B''$.

*Conic.*

(7) If a variable tangent $p$ to a conic meet the tangents $a$ and $b$, drawn at the ends $A$ and $B$ of a diameter, in $A'$ and $B'$, then the product $AA'.BB'$ is constant and equal to the square of the semi-diameter parallel to the tangents.

Let a second tangent $p'$ meet $a$ and $b$ in $A''$, $B''$. Then $(a.pp'ab)=(b.pp'ab)$. Therefore $(A'A''A\infty)=(B'B''\infty B)$. Hence $AA'.BB'=AA''.BB''=$ a constant.

(8) If four points $A$, $B$, $C$, $D$ on a conic be joined to a point $P$ by lines which meet the conic again in $A'$, $B'$, $C'$, $D'$ respectively, then $AB.C'D'$, $AC.B'D'$, $AD.B'C'$, $BC.A'D'$, $BD.A'C'$ and $DC.A'B'$ are collinear with $P$.

Let $AB.C'D'$ be $N$ and $BD.A'C'$ be $K$. If $N$, $P$, $K$ are collinear $(C'.D'CA'B)$ must equal $(B.AB'DC')$. But $(AB'DC')=(A'BD'C)=(D'CA'B)$.

(9) $A$, $B$, $C$, $D$ are four points on a conic, $P$ any point on the conic and $O$ a point not on the conic. $OA$, $OB$ cut the conic again at $A'$, $B'$. Prove that

$$(O.ABCD)=(P.ABCD)\times(P.A'B'CD).$$

Let $OA$, $OB$ and $A'B$ meet $DC$ in $A''$, $B''$, $B'''$ respectively. Then

$$(O.ABCD)=(A''B''CD)=(DCB''A''),$$
$$(P.ABCD)=(A'.ABCD)=(A''B'''CD)=(DCB'''A''),$$
$$(P.A'B'CD)=(B.A'B'CD)=(B'''B''CD)=(DCB''B'''),$$
$$\therefore\ (P.ABCD)(P.A'B'CD)=(DCB'''A'')(DCB''B''')=(DCB''A'')$$
$$=(O.ABCD).$$

(10) If $POP'$, $QOQ'$, $ROR'$, $SOS'$ be four concurrent chords of the same conic, and conics be drawn through $O$, $P$, $Q$, $R$, $S$ and $O$, $P'$, $Q'$, $R'$, $S'$ respectively, prove that these conics touch at $O$.

Let $OT$, $OT'$ be the respective tangents to the two conics at $O$. The tangents $OT$ and $OT'$ will coincide if $(TQRS)=(T'Q'R'S')$, that is if $(O.OQRS)=(O.OQ'R'S')$; that is if $(P.OQRS)=(P'.OQ'R'S')$; that is if $(P'QRS)=(PQ'R'S')$. Since $PP'$, $QQ'$, $RR'$, $SS'$ are conjugate points of an involution this condition is satisfied.

(11) Prove that, if three pencils of rays are projective with one another, the three conics generated by the intersections of corresponding rays of the pencils taken two by two have three common points.

Let $S_1$, $S_2$, $S_3$ be the vertices of the pencils. The two conics determined by the pencils $S_1$, $S_2$ and by the pencils $S_2$, $S_3$ will intersect in $S_2$ and in three other points. If $d_1$, $d_2$, $d_3$ be the rays of the pencils which meet at one of these points, the points $d_1d_2$ and $d_2d_3$ coincide. Therefore $d_1$, $d_2$, $d_3$ intersect at the same point and the conic determined by the pencils $S_1$, $S_3$ passes through this point.

## 103. Involution and Pole and Polar.

### Particular Cases.

A conic determines on every line in its plane an involution, the conjugate points of which are such that the polar of each passes through the other. The double points of the involution are the points in which the line meets the conic. (Art. 95 ($k$) and Art. 77.)

Consider a diameter of a parabola. One of the double points is at infinity; therefore the other bisects the distance between every pair of conjugate points. Hence, if the tangents at the end of a chord $PP'$

of a parabola meet at $T$ and a diameter is drawn through $T$ to meet the chord in $V$ and the parabola in $Q$, $TQ = QV$. Again, if $P$ be any point and $p$ its polar with regard to a hyperbola, and a line be drawn through $P$ parallel to an asymptote to meet the curve in $Q$, and $p$ in $V$, then $PQ = QV$.

**Deductions.**

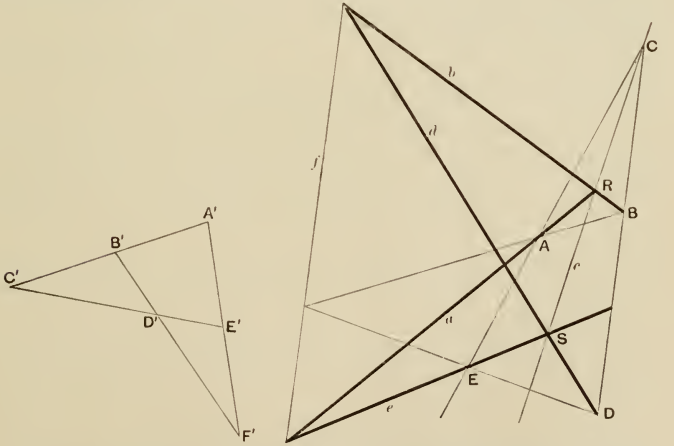
(*a*) *The lines joining the vertices of a triangle to the corresponding vertices of its polar or conjugate triangle meet in a point.*

Let $ABC$ be the triangle and $A'B'C'$ its conjugate. Join $CC'$ to meet $AB$ in $D$ and $A'B'$ in $D'$. Let $AB \,.\, A'B'$ be $S$. Then the polars of $A'$, $B'$, $D'$ and $S$ are $BC$, $AC$, $SC$ and $DC$ respectively. Therefore $(A'B'D'S) = (BASD) = (ABDS)$. Hence the ranges $A'B'D'S$ and $ABDS$ are projective and, since they have a self-corresponding point in $S$, they are in perspective. Therefore $AA'$, $BB'$, $CC'$ are concurrent.

The correlative theorem is.

*The points of intersection of the sides of a triangle with the corresponding sides of its conjugate triangle are collinear.*

(*b*) *If two quadrilaterals are such that five of the vertices of the one are conjugates of five of the vertices of the other with respect to a conic, then the remaining vertices are also conjugate points.*

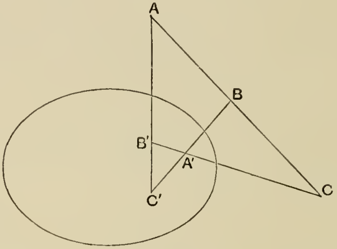


Let $A'$, $B'$, $C'$, $D'$, $E'$, $F'$ be the vertices of one of the quadrilaterals, the points $A'$, $B'$, $C'$, and also the points $C'$, $D'$, $E'$, being collinear. Construct $a$, $b$, $c$, $d$, $e$ the polars of $A'$, $B'$, $C'$, $D'$, $E'$. These pass through two points $R$ and $S$ the poles of $A'B'C'$ and of $C'D'E'$ respectively, and $c$ is the line joining $RS$. The polar of $F'$ is the line joining $a \,.\, e$ to $b \,.\, d$. Denote this line by $f$. If a point $C$ be taken anywhere on $c$ and lines $CAE$ and $CBD$ be drawn to meet $a$, $b$, $c$, $d$ in $A$, $B$, $C$, $D$, then $ABCDE$ has the given relationship to $A'B'C'D'E'$. The theorem will therefore be true, if $f$ passes through the point of intersection of $AB$ and $DE$. This is the case since the triangles $ARB$ and $ESD$ are in perspective.

The correlative theorem is

*If two quadrangles are such that five of the sides of the one are conjugate lines of five of the sides of the other with respect to a conic, then the remaining sides are also conjugate lines.*

(*c*) *If the extremities of each of two diagonals of a complete quadrilateral are conjugate points with respect to a given conic, the extremities of the third diagonal are also conjugate points with respect to the same conic.*



Let $A$, $A'$ and $B$, $B'$ be the ends of the two diagonals which are conjugate points. Let $AB \,.\, A'B'$ be $C$ and $AB' \,.\, BA'$ be $C'$.

Consider the polar triangle of $A'B'C'$. This triangle is in perspective with $A'B'C'$. But the polar of $A'$ passes through $A$, and the polar of $B'$ passes through $B$. Hence $AB$ is the axis of perspective of the two triangles and therefore the polar of $C'$ passes through $C$. Hence the theorem is true.

If $P$ be the pole of $ABC$ with regard to the conic, the theorem may at once be proved from the involution property of the quadrangle $A'B'C'P$.

(*d*) *If four conics are described through three fixed points the lines joining their six other points of intersection to any one of these three points form a pencil in involution.*

Let the three fixed points of intersection be $A$, $B$, $C$ and let the conics be (1), (2), (3), (4). Denote the point of intersection of (1) and (2) by $P_{12}$, that of (2) and (3) by $P_{23}$, and so on.

Let the tangents to (2) at $A$ and $C$ meet at $T_2$. Let $T_2B$ meet the conic (2) in $Q_2$. Then $(ACBQ_2)$ is a harmonic range on the conic (2). Therefore the pencil $(P_{12} \,.\, ACBQ_2)$ is harmonic.

Similarly, if $Q_2P_{12}$ meet (1) in $Q_1$ the pencil $(P_{12} \,.\, ACBQ_1)$ is harmonic.

Similarly, through $P_{23}$, $P_{34}$, $P_{13}$, $P_{14}$, $P_{24}$ draw chords which are harmonic conjugates of the connectors of these points to $B$ with respect to their connectors to $A$ and $C$.

Since there is only one harmonic conjugate of $B$ with respect to $AC$ on each of these conics, these lines must pass three by three through four points $Q_1$, $Q_2$, $Q_3$, $Q_4$.

Hence the points $P_{12}$, $P_{23}$, ... lie on the sides of the quadrangle $Q_1Q_2Q_3Q_4$ and are such that their connectors to $B$ are harmonic conjugates of the sides on which they lie with respect to their connectors to $A$ and $C$.

Hence, if $BP_{12}$, $BP_{23}$, ... meet $AC$ in $B_{12}$, $B_{23}$, ..., $B_{12}$, $B_{23}$, ... are harmonic conjugates with respect to $A$ and $C$ of the points, in which the sides of the quadrangle $Q_1Q_2Q_3Q_4$ meet $AC$. But the points where the sides of $Q_1Q_2Q_3Q_4$ meet $AC$ form an involution. Therefore $B_{12}$, $B_{23}$, $B_{34}$, ... form an involution. Hence their connectors to $B$ form an involution.

(*e*) *The poles of a pencil of lines, which pass through a fixed point, with respect to a fixed triangle, lie on a conic which circumscribes the triangle, and the tangents to this conic at the vertices of the triangle intersect the opposite sides in the points, where they are met by the polar of the given point.*

Let the variable line $p$ pass through a fixed point $O$ and meet the sides of the triangle $ABC$ in $A'$, $B'$, $C'$. Let $A''$, $B''$, $C''$ be the harmonic conjugates of $A'$, $B'$, $C'$ with respect to the vertices of the triangle with which they are collinear. Then $P$ the pole of $p$ is the point of intersection of $AA''$, $BB''$, $CC''$. For different positions of $p$ the ranges $A'$ and $C'$ are projective. These ranges are projective with those described by $A''$ and $C''$, which are therefore projective with each other. Hence the pencils $AA''$ and $CC''$ are projective and therefore the locus of $P$ is a conic through $A$ and $C$, which by symmetry also passes through $B$.

The tangent to the conic at $A$ may be constructed by joining $O$ to $A$ to meet $BC$ in $A_1$ and taking $A_1'$ the harmonic conjugate of $A_1$ with respect to $BC$, in which case the line $AA_1'$ is the tangent at $A$. Hence the connector of the points, where the tangents at $A$, $B$ and $C$ meet the opposite sides of the triangle, is the polar of $O$.

Conversely the following theorem is obtained :

*If $P$ be any point on a conic circumscribing a triangle $ABC$, the polar of $P$, with respect to the triangle $ABC$, passes through the pole, with respect to the conic, of the line joining the points, where the tangents at $A$, $B$ and $C$ to the conic meet the opposite sides of the triangle $ABC$.*

## EXAMPLES.

(1) $A$, $A'$, $B$, $B'$ are any four points on a conic. Find the points $P$, $P'$ on the conic separating harmonically $A$ and $A'$ and also $B$ and $B'$ ; and show that the line joining the points is the same for all conics passing through the four points.

The line determining these points is the polar of $AA' . BB'$.

(2) $AA'$, $BB'$, $CC'$ are three pairs of points homographically related on a conic. If $O$, $O'$ be the self-conjugate points of the system, show that the pole of $OO'$ and the points $OB . O'A'$ and $OA . O'B'$ are collinear.

From the data $(O . OO'BA)=(O' . OO'B'A')=(O' . O'OA'B')$.

Hence the pencils $(O . OO'BA)$ and $(O' . O'OA'B')$ are projective ; but $O'O$ is a self-corresponding ray. Therefore the pencils are in perspective. Hence the given points are collinear.

(3) Two lines $SP$ and $SQ$ at right angles are drawn through the focus of a conic, intersecting the directrix in $P$ and $Q$ : $R$ is any point on the conic and $PR$ and $QR$ meet the conic again in $L$ and $M$. Prove that $LM$ is a focal chord.

$PS$ and $QS$ are conjugate lines (Art. 96) and therefore $P$ and $Q$ are conjugate points. Hence by the converse of the construction for conjugate points (Art. 95 (*j*)) the chord $LM$ must pass through the pole of $PQ$, that is through the focus $S$.

(4) $PQ$ is a chord of a conic. $O$ is a point on $PQ$ and $M$ is a point on the polar of $O$. A parallel through $O$ to $MQ$ cuts $MP$ at $B$ and the polar at $A$. Show that $B$ bisects $OA$.

Let $MA$ meet $PQ$ in $T$. Then $(TOPQ)$ is harmonic. Therefore the pencil $(M.TOPQ)$ is harmonic. Taking intercepts on $OBA$, it is seen that $OB=BA$.

(5) $M$, $N$ are a fixed pair of conjugate points with regard to a conic, $O$ a variable point on the conic; $OM$, $ON$ meet the conic again in $P$, $Q$ respectively; prove that $P$, $Q$ are a pair of conjugate points of a definite involution.

$PQ$ passes through the pole of the line $MN$ which is a fixed point.

(6) The sides $BC$, $CA$, $AB$ of a triangle $ABC$, self-conjugate with respect to a given conic, intersect any line $p$ in $A'$, $B'$, $C'$; also $O$ is the pole of $p$. Prove that $OA$, $OA'$; $OB$, $OB'$; $OC$, $OC'$ form a pencil in involution.

$OA$ is the polar of $A'$, $OB$ of $B'$, $OC$ of $C'$. Hence the theorem is true.

(7) Given two points $A$ and $B$ on a conic, find two other points $P$ and $Q$ on the conic such that $A$ and $B$ shall lie on a circle of which $PQ$ is a diameter.

Draw through $A$ two pairs of orthogonal rays $AC$, $AC'$ and $AD$, $AD'$. Join $CC'$, $DD'$ to meet in $K$. Every chord through $K$ will determine on the conic a pair of conjugate points of an involution which subtend a right angle at $A$. Similarly from $B$ a point $L$ can be found such that every chord through $L$ will subtend a right angle at $B$. Hence the line $KL$ meets the conic in the required points $P$ and $Q$.

(8) Through a given point in the plane of a conic draw a chord such that it subtends a right angle at a given point on the conic. Show that in general there is one and only one solution.

Describe on the conic the involution formed by pairs of lines at right angles through the given point on the conic. Let the chords which determine this involution pass through $O$. Join $O$ to the given point, which is not on the conic. The line so obtained determines the required chord.

(9) The anharmonic ratio of four diameters of a conic is equal to that of their four conjugates.

This is a particular case of Art. 95 ($l$). The diameter $a'$ conjugate to a diameter $a$ is obtained by joining the pole of $a$, which is at infinity, to the centre of the conic. The theorem may also be proved from the fact that conjugate diameters are parallel to supplemental chords.

(10) Find the locus of the centre of a conic which circumscribes a given quadrangle.

Draw diameters through the middle points of the sides. These form a pencil whose anharmonic ratio is equal to that of their conjugate diameters. The anharmonic ratio of these lines is given, since the conjugate diameters are parallel to the sides of the quadrangle. Hence the locus is a conic through the middle points of the sides.

(11) $A, A'$; $B, B'$; $C, C'$ are three pairs of points on a conic. The double points of the involutions determined by $(B, B'; C, C')$, $(C, C'; A, A')$, $(A, A'; B, B')$ are $P, P'$; $Q, Q'$; $R, R'$ respectively. Show that the involutions determined by $(A, A'; P, P')$, $(B, B'; Q, Q')$, $(C, C'; R, R')$ have a common pair of conjugate points.

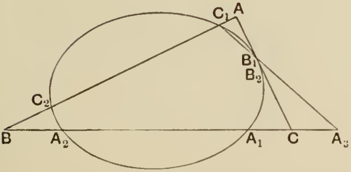
If $S, T, U$ are the vertices of the triangle formed by $AA'$, $BB'$, $CC'$, this triangle has $PP'$, $QQ'$, $RR'$ for its polar triangle and therefore the sides of $STU$ meet the corresponding sides of $PP'$, $QQ'$, $RR'$ in three collinear points. The line on which these points are situated determines a common pair of conjugates of the three involutions.

(12) If a triangle is inscribed in a conic, the points in which two of the sides are cut by any straight line drawn through the pole of the third side are a pair of conjugate points with respect to the conic.

Draw $p$ a chord through the pole $R$ of the side $BC$ of the inscribed triangle $ABC$ to meet the conic in $P$ and $P'$, and the sides $AB$ and $AC$ in $Q$ and $Q'$. Then the range $(BCPP')$ is harmonic. Project this range from $A$ upon $p$. Then $(QQ'PP')$ is harmonic and therefore $Q$ and $Q'$ are conjugate points. Thus it may be shown that the connectors of four pairs of conjugate points on $AB$ and $AC$ pass through $R$ and therefore, since these pairs of conjugate points form two projective ranges, the connectors of all such pairs pass through $R$.

## 104. Carnot's Theorem.

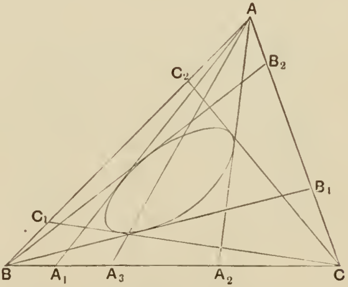
### Particular Cases.



Carnot's Theorem (Art. 99) may be put into a slightly different form.

If in the figure $C_1B_1$ meet $BC$ in $A_3$ then by Menelaus' Theorem

$$\frac{AC_1}{BC_1}\cdot\frac{CB_1}{AB_1}=\frac{1}{\frac{BA_3}{CA_3}}.$$



The correlative of Carnot's Theorem (Art. 99) may be put into a slightly different form.

If in the figure the line joining $A$ to $BB_1 . CC_1$ meet $BC$ in $A_3$ then by Ceva's Theorem

$$\frac{AC_1}{BC_1}\cdot\frac{CB_1}{AB_1}=-\frac{1}{\frac{BA_3}{CA_3}}.$$

Hence the relation

$$\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2}\cdot\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}=1$$

becomes

$$\frac{AC_2}{BC_2}\cdot\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_2}{AB_2}=\frac{BA_3}{CA_3}.$$

(i) If $A$ be situated on the conic, $C_1B_1$ becomes the tangent at $A$. Hence the following is obtained:

*If one vertex $A$ of a triangle $ABC$ be on a conic and the sides $AB$ and $AC$ meet the conic in $C_2$, $B_2$ and the side $BC$ meet the conic in $A_1$, $A_2$ and the tangent at $A$ in $A_3$, then*

$$\frac{AC_2}{BC_2}\cdot\frac{CB_2}{AB_2}\cdot\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}=\frac{BA_3}{CA_3}$$

*or*

$$\frac{AC_2}{BC_2}\cdot\frac{CB_2}{AB_2}\cdot\frac{BA_2}{CA_2}=(BCA_3A_1).$$

(ii) If the three vertices of the triangle are all situated on the conic and the tangents at these points meet the opposite sides in $A_3$, $B_3$, $C_3$ the theorem becomes

$$\frac{BA_3}{CA_3}\cdot\frac{CB_3}{AB_3}\cdot\frac{AC_3}{BC_3}=1.$$

Hence $A_3$, $B_3$, $C_3$ are collinear.

(iii) If the conic break up into a pair of lines the theorem is still true.

If one of the lines is the line at infinity, Menelaus' theorem is obtained, viz.

$$\frac{AC_1}{BC_1}\cdot\frac{BA_1}{CA_1}\cdot\frac{CB_1}{AB_1}=1.$$

Hence the relation

$$\frac{AC_1}{BC_1}\cdot\frac{AC_2}{BC_2}\cdot\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_1}{AB_1}\cdot\frac{CB_2}{AB_2}=1$$

becomes

$$\frac{AC_2}{BC_2}\cdot\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}\cdot\frac{CB_2}{AB_2}=-\frac{BA_3}{CA_3}.$$

If $BC$ be a tangent to the conic, $A_3$ becomes the point of contact. Hence the following is obtained:

*If one side $BC$ of a triangle $ABC$ touch a conic at $A_3$ and the tangents from $B$ and $C$ meet the opposite sides in $B_2$, $C_2$ and the tangent from $A$ meet $BC$ in $A_1$, $A_2$, then*

$$\frac{AC_2}{BC_2}\cdot\frac{CB_2}{AB_2}\cdot\frac{BA_1}{CA_1}\cdot\frac{BA_2}{CA_2}=-\frac{BA_3}{CA_3}$$

*or*

$$-\frac{AC_2}{BC_2}\cdot\frac{CB_2}{AB_2}\cdot\frac{BA_2}{CA_2}=(BCA_3A_1).$$

If the three sides of the triangle are all tangents to the conic and their points of contact are $A_3$, $B_3$, $C_3$ the theorem becomes

$$\frac{BA_3}{CA_3}\cdot\frac{CB_3}{AB_3}\cdot\frac{AC_3}{BC_3}=-1.$$

Hence $AA_3$, $BB_3$, $CC_3$ are concurrent.

If the conic break up into a pair of points the theorem is still true.

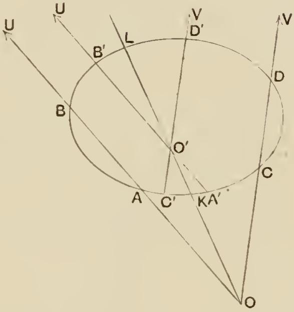
If one of the points is the centroid of the triangle, Ceva's theorem is obtained, viz.

$$\frac{AC_1}{BC_1}\cdot\frac{BA_1}{CA_1}\cdot\frac{CB_1}{AB_1}=-1.$$

**Deductions.**

(*a*) **Newton's Theorem.** *If through any two points $O$ and $O'$ two pairs of parallel chords $OAB$, $OCD$* and *$O'A'B'$, $O'C'D'$ are drawn to meet a conic in $A$, $B$, $C$, $D$ and $A'$, $B'$, $C'$, $D'$, then*

$$\frac{OA \cdot OB}{OC \cdot OD}=\frac{O'A' \cdot O'B'}{O'C' \cdot O'D'}.$$



The lines $OAB$, $O'A'B'$ meet at a point at infinity $U$, and the lines $OCD$, $O'C'D'$ at a point at infinity $V$.

Let $OO'$ meet the conic in $K$ and $L$.

From the triangle $OO'U$

$$\frac{OA \cdot OB}{UA \cdot UB} \cdot \frac{UA' \cdot UB'}{O'A' \cdot O'B'} \cdot \frac{O'L \cdot O'K}{OL \cdot OK}=1.$$

But $\dfrac{UA' \cdot UB'}{UA \cdot UB}=1$, since $U$ is at infinity.

Therefore $OA \cdot OB=O'A' \cdot O'B' \dfrac{OL \cdot OK}{O'L \cdot O'K}$.

Similarly, from the triangle $OO'V$,

$$OC \cdot OD=O'C' \cdot O'D' \frac{OL \cdot OK}{O'L \cdot O'K}.$$

Therefore $$\frac{OA \cdot OB}{OC \cdot OD}=\frac{O'A' \cdot O'B'}{O'C' \cdot O'D'}.$$

The above proof assumes that the line $OO'$ meets the conic in a pair of real points $K$ and $L$. If $OO'$ does not meet the conic in real points, $O$ and $O'$ must both be external to the conic. Take any point $O''$ within the conic. Then the theorem holds for the pairs of parallel chords through $O$ and $O''$ and for pairs of parallel chords through $O'$ and $O''$. Hence it holds for parallel chords through $O$ and $O'$.

*Deductions from Newton's Theorem.*

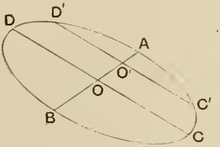
(1) If $d$ and $d'$ are the semi-diameters, and $t$ and $t'$ the tangents, parallel to $OAB$ and $OCD$ respectively, then

$$\frac{OA \cdot OB}{OC \cdot OD}=\frac{d^2}{d'^2}=\frac{t^2}{t'^2}.$$

(2) In the case of an Ellipse, if $O$ be the centre of the curve, $OA$ and $OC$ a pair of conjugate diameters, and $O'$ be taken on $OA$, then in the figure

$$\frac{OD^2}{OA^2}=\frac{O'D'^2}{BO' \cdot O'A}=\frac{O'D'^2}{OA^2-OO'^2};$$

$$\therefore \quad \frac{O'D'^2}{OD^2}+\frac{OO'^2}{OA^2}=1.$$



(3) In the case of a Parabola, let $UMU'$ be an ordinate to a diameter $PM$, and $S$ the focus. Draw the focal chord $QQ'$ parallel to $UU'$. Then by

Newton's Theorem, since the diameter meets the curve at infinity,

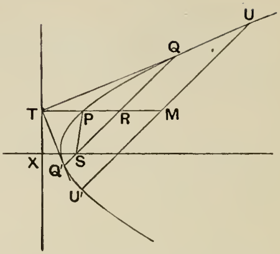
$$\frac{UM \, . \, U'M}{PM} = \frac{QR \, . \, Q'R}{PR} .$$

But in the figure, if $XT$ be the directrix, $RP = PT = SP$. Also, since $QTQ'$ is a right angle, $R$ is centre of circle through $TQQ'$,

$$\therefore \; QR^2 = RT^2 = 4 \, . \, SP^2,$$

$$\therefore \; \frac{QR \, . \, Q'R}{PR} = 4 \, . \, SP,$$

$$\therefore \; UM^2 = 4 \, . \, SP \, . \, PM.$$



(*b*) *If a circle and a central conic intersect in four points, the pairs of chords of intersection are equally inclined to the axes of the conic.*

Let the conic and the circle intersect in $A$, $B$, $C$, $D$ and let the chords $AB$ and $CD$ intersect in $O$. Let $d$ and $d'$ be the semi-diameters of the conic parallel to $AB$ and $CD$ respectively.

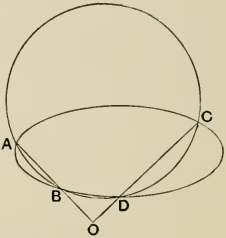
Then from the conic

$$\frac{OA \, . \, OB}{OC \, . \, OD} = \frac{d^2}{d'^2},$$

and from the circle

$$\frac{OA \, . \, OB}{OC \, . \, OD} = 1.$$

Therefore $\quad \frac{d^2}{d'^2} = 1$ and $d = d'$.



Hence the semi-diameters of the conic parallel to $AB$ and $CD$ are equal, and therefore equally inclined to the axes of the conic, as are also the chords of intersection of the circle and conic.

*Centre of Curvature.* If a circle be described to meet a conic in three coincident points $P$, the circle is termed *the circle of curvature* at $P$ and its centre *the centre of curvature.* From the preceding it is seen that the tangent to the conic at $P$ and the common chord of the circle and conic are equally inclined to the axes of the conic. The centre of curvature is obviously on the normal at $P$. Hence the following construction is obtained for the centre of curvature at any point $P$.

Draw $PT$ the tangent at $P$ and $PL$ a chord equally inclined to the axis. Let $M$ be the middle point of $PL$, then $C$ the point of intersection of the normal at $P$ and the perpendicular through $M$ to $PL$ is the centre of curvature.

## EXAMPLES.

(1) If $A_1$, $B_1$, $C_1$ be the points where three concurrent lines through $A$, $B$, $C$ meet the opposite sides of a triangle, and a conic through $A_1$, $B_1$, $C_1$ meet the sides in $A_2$, $B_2$, $C_2$, then the lines $AA_2$, $BB_2$, $CC_2$ are concurrent.

(2) If tangents are drawn to a conic from the vertices of a triangle to meet the opposite sides in $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$, then a conic can be described through these six points.

(3) If a conic cut the sides of a triangle $ABC$ in $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$, then a conic can be described to touch the six lines $AA_1$, $AA_2$, $BB_1$, $BB_2$, $CC_1$, $CC_2$.

(4) If a conic touch the sides of a triangle $ABC$ at $A'$, $B'$, $C'$, and $A''$, $B''$ be the harmonic conjugates of $A'$ and $B'$ with respect to $BC$ and $CA$, a conic can be described to touch the sides of the triangle at $A''$, $B''$, $C'$.

(5) If a straight line meet a hyperbola in $A$ and $B$, and its asymptotes in $A'$ and $B'$, then $AA'$ and $BB'$ are equal.

(6) If a conic meet the sides of a triangle $ABC$ in $A_1$, $A_2$, $B_1$, $B_2$, $C_1$, $C_2$, and $A_1B_1$, $A_2B_2$ intersect in $C_3$, $B_1C_1$ and $B_2C_2$ in $A_3$, $C_1A_1$ and $C_2A_2$ in $B_3$, then the lines $AA_3$, $BB_3$, $CC_3$ are concurrent.

(7) The vertices $A$, $B$, $C$ of a triangle are joined to any point $P$, and the lines $PA$, $PB$, $PC$ meet the opposite sides in $A_1$, $B_1$, $C_1$. A conic is described through $A$, $A_1$, $B_1$, $C_1$, meeting $BC$ in $A_2$, and the tangent to the conic at $A$ meets $BC$ in $A_3$. Prove that the range $BCA_2A_3$ is harmonic and that $A_2$, $A_3$ are conjugate points of a definite involution.

(8) If $P$, $Q$, $R$, $S$ be the points of contact of the sides $AB$, $BC$, $CD$, $DA$ of a quadrilateral circumscribed to a conic,

$$\frac{AP}{BP}\cdot\frac{BQ}{CQ}\cdot\frac{CR}{DR}\cdot\frac{DS}{AS}=1.$$

(9) Extend Carnot's theorem to the case of a polygon meeting the sides of a conic.

(10) If a hexagon $ABCDEF$ circumscribe a conic and the sides $AB$, $BC$, $CD$, $DE$, $EF$, $FA$ touch the conic at $A'$, $B'$, $C'$, $D'$, $E'$, $F'$, and the ratios

$$\frac{AA'}{BA'},\ \frac{BB'}{CB'} \text{ etc. be denoted by } a, b, c, d, e, f,$$

prove that

$$abcdef=1.$$

(11) Show that, if from any two points the vertices of a triangle are projected upon the opposite sides, the six points so obtained lie on a conic, and find by the ruler only the points in which it cuts again the rays of projection.

(12) Through a point $P$ on a conic chords $PA$, $PB$ are drawn, and chords $MCC'$, $MDD'$ are drawn, respectively parallel to these, through another point $M$, not on the conic. On $PA$, $PB$ lengths $PQ$, $PR$ are taken inversely proportional to

$$\frac{MC.MC'}{PA},\quad \frac{MD.MD'}{PB}.$$

Prove that the normal at $P$ to the conic is a diameter of the circle $PQR$.

(13) Prove that, if a circle intersects a parabola in four points, the pairs of connectors of these points are equally inclined to the axis of the parabola, and obtain a construction for the centre of curvature of any point on the parabola.

# CHAPTER XVI

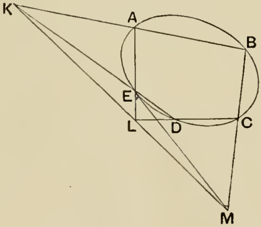
## DEDUCTIONS FROM PASCAL'S THEOREM, DESARGUES' THEOREM, AND THE COMPLETE INSCRIBED QUADRANGLE AND CIRCUMSCRIBED QUADRILATERAL OF A CONIC. CONICS IN SELF-PERSPECTIVE. THE RECTANGULAR HYPERBOLA

### 105. Pascal's and Brianchon's Theorems.

**Particular Cases.**

Pascal's and Brianchon's Theorems are true for any six points on any conic and for any six tangents to any conic. There are certain particular cases which should be noted.

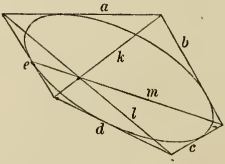
(i) If two of the six points coincide, one of the sides of the hexagon is replaced by a tangent. Hence the following is obtained:



*If ABCDE be a pentagon inscribed in a conic and BC meets the connector of AB.DE (≡ K) and CD.EA (≡ L) in M, then the line ME is the tangent at E.*

Hence given any five points on a conic the tangents at any of these points can be constructed.

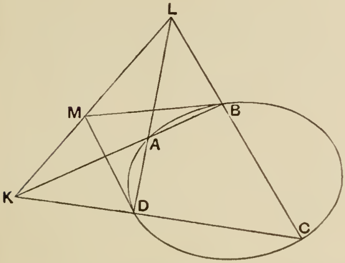
If two of the six tangents coincide, one of the vertices of the hexagon is replaced by the point of contact of a tangent. Hence the following is obtained:



*If abcde be a pentagon circumscribed to a conic and m is the connector of bc to the point of intersection of ab.de (≡ k) and cd.ea (≡ l), then m passes through the point of contact of e.*

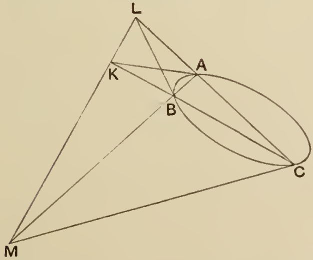
Hence given any five tangents to a conic the points of contact of any of these tangents can be constructed.

(ii) If two pairs of the six points coincide, the properties of the complete quadrangle inscribed in a conic are obtained, viz.



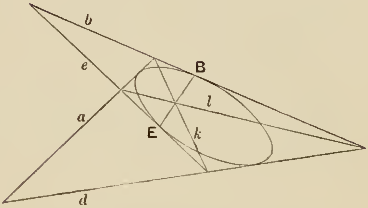
*If a quadrangle be inscribed in a conic, the points of intersection of the tangents at the vertices lie two by two on the sides of the diagonal points triangle of the quadrangle.*

(iii) If three pairs of the six points coincide, the following theorem is obtained:



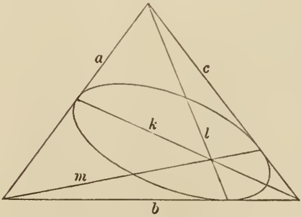
*If a triangle be inscribed in a conic, the tangents at the vertices intersect the opposite sides in three collinear points.*

If two pairs of the six tangents coincide, the properties of the complete quadrilateral circumscribed to a conic are obtained, viz.



*If a quadrilateral be circumscribed to a conic, the connectors of the points of contact of the tangents pass two by two through the points of intersection of the diagonals of the quadrilateral.*

If three pairs of the six tangents coincide, the following theorem is obtained:



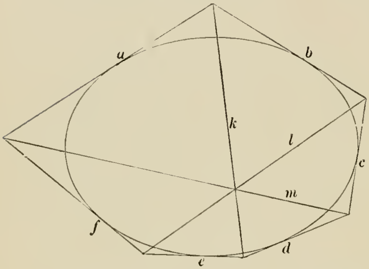
*If a triangle be circumscribed to a conic, the lines joining the points of contact to the opposite vertices are concurrent.*

*The Parabola.* In the case of the parabola the point at infinity on any diameter is on the curve and the line at infinity is a tangent to the curve. Consequently the theorems of this article may be expressed in a simpler form for the parabola. The deduction of these results is left as an exercise for the reader.

*Pascal's Hexagons for six points.* If any six points are taken on a conic, 60 Pascal Hexagons can be formed by taking the six points in different orders. The properties of such hexagons are given in Art 146.

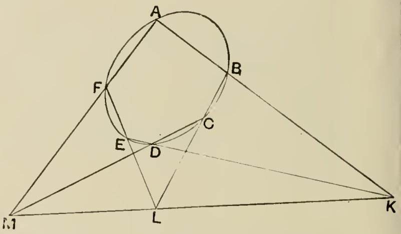
**Deductions.**

(*a*) *The locus of the poles of tangents to one conic with respect to a second conic is a conic.*



Let *a*, *b*, *c*, *d*, *e*, *f* be any six tangents to the given conic. Then by Brianchon's theorem, *k*, *l*, *m* the three connectors of pairs of opposite vertices of the hexagon formed by these lines are concurrent. Let *A*, *B*, *C*, *D*, *E*, *F* be the poles of *a*, *b*, *c*, *d*, *e*, *f* with respect to the second conic. Then, since *k*, *l*, *m* are concurrent, *K*, *L*, *M* the points of intersection of pairs of opposite sides of the hexagon *ABCDEF* are collinear. Hence by the converse of Pascal's theorem the points *A*, *B*, *C*, *D*, *E*, *F* lie on a conic. If *f* be regarded as a

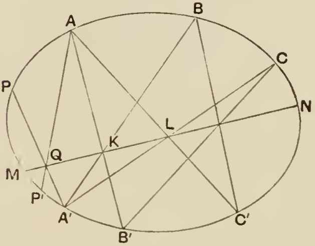
*The envelope of the polars of points on one conic with respect to a second conic is a conic.*



Let *A*, *B*, *C*, *D*, *E*, *F* be any six points on the given conic. Then by Pascal's theorem, *K*, *L*, *M* the three points of intersection of pairs of opposite sides of the hexagon formed by these points are collinear. Let *a*, *b*, *c*, *d*, *e*, *f* be the polars of *A*, *B*, *C*, *D*, *E*, *F* with respect to the second conic. Then, since *K*, *L*, *M* are collinear, *k*, *l*, *m* the connectors of pairs of opposite vertices of the hexagon *abcdef* are concurrent. Hence by the converse of Brianchon's theorem the lines *a*, *b*, *c*, *d*, *e*, *f* touch a conic. If *F* be regarded variable line, it follows that the locus of its pole $F$ is the conic $ABCDE$.

as a variable point, it follows that the envelope of its polar $f$ is the conic $abcde$.

(*b*) *Construction of Projective Ranges on a Conic and of their self-corresponding points.*



Two projective ranges on a conic are determined, when three pairs of corresponding elements are given. By Pascal's theorem the self-corresponding elements can be found.

Let $A, B, C, A', B', C'$ be the determining elements. Then the pencils $(A\,.\,A'B'C')$ and $(A'\,.\,ABC)$ have a self-corresponding ray in $AA'$ and are therefore in perspective. The axis of perspective is the Pascal line of the hexagon $AC'BA'CB'$, that is the line $KL$ in the figure. If $Q$ be any point on $KL$, the lines $AQ$ and $A'Q$ meet the conic in points $P$ and $P'$, which are corresponding points of the ranges $ABC$ and $A'B'C'$. If the line $KL$ meet the conic in $M$ and $N$, these points correspond to themselves and are the two self-corresponding points of the ranges $ABC$ and $A'B'C'$ on the conic.

Given the self-corresponding points, $M$ and $N$, of two projective ranges, and a pair of corresponding points, $A$ and $A'$, the ranges can be constructed. Take any point $Q$ on $MN$ and let $QA'$ and $QA$ meet the conic in $P$ and $P'$ respectively; then $P$ and $P'$ are corresponding points.

Conversely, if points $Q$ on a fixed line $MN$ be joined to any pair of fixed points $A$ and $A'$, and $QA$, $QA'$ meet the conic in $P'$ and $P$, these points are corresponding points of two ranges having $M$ and $N$ for self-corresponding points and $A$ and $A'$ for corresponding points.

By the correlative method the self-corresponding elements of two projective systems of tangents may be constructed.

## EXAMPLES.

(1) If through the points $A$ and $B$, where a tangent to any hyperbola meets the asymptotes, lines parallel to the asymptotes be drawn to intersect in $Q$, and $O$ be the centre of the hyperbola, $OQ$ passes through the point of contact of the tangent and is bisected at that point.

This is a particular case of (iii).

(2) If through the points of contact of two tangents to a parabola, which intersect in $T$, lines parallel to these tangents be drawn to intersect in $V$, $TV$ is a diameter of the parabola and it is bisected by the chord of contact.

This is a particular case of (iii).

(3) $P$ and $Q$ are two points on a conic and pairs of chords $PA$, $PB$ are drawn through $P$, and $QC$, $QD$ through $Q$. Show that if $PA$ and $QC$ meet in $R$ and $PB$ and $QD$ in $T$, then the lines $AD$, $BC$, $RT$ are concurrent.

$PACQDB$ is a hexagon inscribed in a conic and therefore by Pascal's theorem the triangles $ACR$ and $DBT$ are in perspective.

(4) Three straight lines are drawn parallel to the sides of a triangle; show that their six points of intersection with the sides of the triangle lie on a conic.

Converse of Pascal's theorem with the line at infinity for axis of perspective.

(5) The hexagon which is formed by the six points of intersection of alternate pairs of sides of a Pascal hexagon is a Brianchon hexagon.

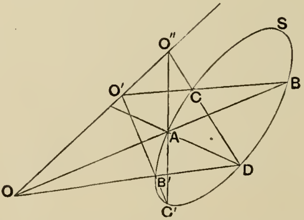
If the vertices of the hexagon formed by the six points of intersection of alternate pairs of sides be $A$, $B$, $C$, $D$, $E$, $F$, the triangles $ACE$ and $BDF$ are in perspective, the axis of perspective being the Pascal line of the given hexagon. Hence $AD$, $BE$, $CF$ are concurrent and the hexagon is a Brianchon hexagon.

(6) If through any two fixed points $C$ and $C'$ on a conic chords $CB'$, $CA'$, $C'A$, $C'B$ be drawn and $CA$, $C'B$ intersect in $W$ and $C'B'$, $CA'$ in $W'$, then $AB'$, $BA'$, $WW'$ are concurrent.

Consider the Pascal hexagon $ACA'BC'B'$.

(7) If the sides of a given triangle meet a conic in six points, the triangle formed by joining the six points in pairs is in perspective with the given triangle.

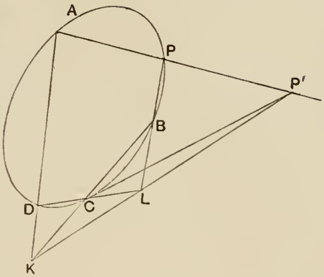
(8) $O$, $O'$, $O''$ are three fixed collinear points and $A$ is any point on a fixed conic $S$. $AO$ meets $S$ in $B$, $BO'$ meets $S$ in $C$, $CO''$ meets $S$ in $D$. Prove that $AD$ will always pass through a fixed point.



Range described by $A$ is projective with
range described by $B$,
hence with
range described by $C$,
hence with
range described by $D$.

Let $OD$ meet the conic in $B'$ and $O'B'$ meet the conic in $C'$. Then $O''C'$ will pass through $A$, for $ABCDB'C'$ is a Pascal hexagon. Therefore $A$ and $D$ mutually correspond. Therefore they are conjugate points of an involution on the conic and consequently $AD$ always passes through a fixed point.

(9) If the conics of a system pass through four points $A$, $B$, $C$, $D$, and a fixed chord through $A$ meet one of the conics in $P$ and the tangent to the same conic at one of the points ($C$) in $P'$, then $P$ and $P'$ describe projective ranges.

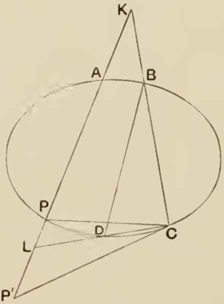


Consider the Pascal hexagon $APBCCD$ in which two vertices are coincident at $C$. Then, in the figure, $K$ is fixed and $DC$ and $AP$ are fixed lines.

The range $P'$ is projective with the range $L$ on $CD$. This is projective with the pencil $BL$ and this is projective with the range $P$.

Therefore the ranges $P$ and $P'$ are projective.

(10) A system of conics passes through four real points $A$, $B$, $C$, $D$. Prove that the harmonic conjugate of $AC$ with regard to $AB$ and $AD$ is cut by the conics and the tangents at $C$ in an involution, whose double points are real.



Let $AP$ the harmonic conjugate of $AC$ with regard to $AB$ and $AD$ meet a conic of the system in $P$ and the tangent at $C$ in $P'$. Let $CB$ and $CD$ meet $AP$ in $K$ and $L$. Then since the pencil $(A\,.\,BDCP)$ is harmonic, $(C\,.\,BDCP)$ is harmonic. Therefore $(KLP'P)$ is harmonic and $P$ and $P'$ are a pair of conjugate points of an involution of which $K$ and $L$ are the double points.

(11) A quadrangle $ABCD$ is inscribed in a parabola: through two of its vertices $C$ and $D$ straight lines are drawn parallel to the axis, meeting $DA$, $BC$ in $P$ and $Q$; show that $PQ$ is parallel to $AB$.

If $\infty$ be the point at infinity on the parabola, the hexagon $BAD\infty\infty C$ is inscribed in the parabola. Therefore the points $BA\,.\,\infty\infty$, $AD\,.\,\infty C$, $CB\,.\,D\infty$ are collinear. Hence $PQ$ and $AB$ are parallel.

## 106. Desargues' Theorem.

### Particular Cases.

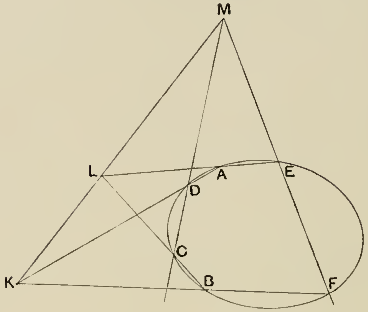
(i) If in Desargues' theorem (Art. 101) the transversal be drawn through one of the four points of intersection of the system of conics, it does not meet the system of conics in an involution properly so called. In this case, however, the following theorems hold:

(*a*) *A system of conics through four fixed points determines on any two transversals, drawn through two of these points, two ranges which are in perspective.*

Let $A$, $B$, $C$, $D$ be the points of intersection of the conics. Through

$A$ and $B$ draw any two transversals $AE$ and $BF$ to meet one of the conics in $E$ and $F$.

Let $AE.CB$ be $L$,
$AD.BF$ be $K$,
and $DC.EF$ be $M$.



Then $K$, $L$, $M$ are collinear by Pascal's theorem. But the points $L$ and $K$ are fixed as is the line $DC$, and therefore $M$ is a fixed point. Hence, since $EF$ passes through the fixed point $M$, the ranges $E$ and $F$ are in perspective.

(*b*) *A system of conics through four fixed points determines two projective ranges on any two transversals, drawn through one of these points.*

If a third transversal be drawn through one of the other points of intersection of the conics, the range determined by the conics on this transversal is in perspective with each of the ranges on the given transversal, which are therefore projective with each other.

From (*a*) and (*b*) the following theorems are obtained:

*Given four points on a conic $A$, $B$, $C$, $D$ and two fixed lines $AE$, $BF$ each passing through one of them, and meeting the conic in $E$ and $F$, the line $EF$ passes through a fixed point.*

*Given four points on a conic $A$, $B$, $C$, $D$ and two fixed lines through one of them, $AE$, $AF$, meeting the conic in $E$ and $F$, the envelope of $EF$ is a conic.*

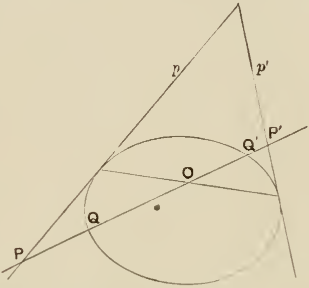
The correlatives of theorems (i) (*a*) and (i) (*b*) are as follows:

*If a system of conics be described to touch four fixed lines, the pencils formed by the tangents to the conics from two points, each on one of the fixed lines, are in perspective.*

*If a system of conics be described to touch four fixed lines, the pencils formed by the tangents to the conics from two points, both on one of the fixed lines, are projective.*

(ii) *If a system of conics be described through two given points, $Q$ and $Q'$, to touch two given lines $p$ and $p'$, the chords of contact of the conics with $p$ and $p'$ pass through one or other of two fixed points, and the poles of $QQ'$ lie on one or other of two fixed lines.*

Consider the quadrangle formed by the points in which a conic of the system meets $p$ and $p'$. Let $QQ'$ meet $p$ and $p'$ in $P$ and $P'$ and their chord of contact in $O$. Then $QQ'$, $PP'$ are two pairs of conjugate points of the involution determined by the conic on $QQ'$. Of this involution $O$ is a double point and, since this is a given involution, $O$ is one or other of two given points.



The second part of the theorem is the correlative of the first.

(iii) *The polar of a given point with respect to a conic passes through the conjugate of the given point with respect to every quadrangle inscribed in the conic.* (See Art. 57.)

Let $O$ be the given point, $ABCD$ any inscribed quadrangle, and $G$ and $E$ the points $AB\,.\,DC$ and $AC\,.\,BD$. Join $A$, $B$, $C$, $D$ to $O$ to meet the conic again in $A'$, $B'$, $C'$, $D'$, respectively. Let $AB$ and $C'D'$ meet in $K$ and let $OK$ meet $DC$ in $K'$ and the polar of $O$ in $M$. Then $OMKK'$ is harmonic and therefore $(G\,.\,OMAC)$ is harmonic.

By Example (8), Art. 102, $BD$ and $C'A'$ meet in a point $N$ on $OK$. Let $AC$ meet $OK$ in $N'$. Then $N$ and $N'$ are harmonic conjugates of $O$ and $M$. Therefore $(E\,.\,OMAD)$ is harmonic. Hence $M$ is the conjugate of $O$ with respect to the quadrangle $ABCD$.

If the connector of the given point $O$ to its conjugate $M$ with regard to the quadrangle meets a conic in real points $P$ and $P'$, the points $O$ and $M$ are double points of the involution determined by conics circumscribing the quadrangle. Therefore $OMPP'$ is harmonic and $O$, $M$ are conjugate points with regard to the conic.

**Deductions.**

(*a*) *If three conics circumscribe the same quadrangle, a common tangent to any two is cut harmonically by the third conic.*

The points of contact are the double points of the involution determined by the conics on the common tangent.

*If through the intersection of a pair of common chords of two conics a tangent be drawn to one of them, this tangent is cut harmonically by the other conic.*

This theorem is a particular case of the preceding, one conic being replaced by a pair of lines. The tangent from the point of intersection of the common chords may be looked upon as a common tangent.

*If two conics have double contact with each other, any tangent to one is cut harmonically, at the points where it meets the other, at its point of contact and at its point of intersection with the common chord.* (Cf. Art. 131.)

The preceding theorem takes this form when the two common chords coincide.

The correlatives of the preceding should be noticed: they are as follows:

*If three conics are inscribed in the same quadrilateral, the tangents to two of them at a point of intersection are harmonic conjugates of the tangents from this point to the third.*

*The chord joining the points of intersection of two pairs of common tangents to two conics, the tangent to one of them at a point where this chord meets it, and the pair of tangents from this point to the other conic, form a harmonic pencil.*

*If two conics have double contact with each other, the tangent at any point* ($P$) *on one is a harmonic conjugate of the line joining its point of contact* ($P$) *to the intersection of the common tangents, with respect to the two tangents from that point* ($P$) *to the other conic.* (Cf. Art. 131.)

(*b*) There is an important group of theorems the proofs of which depend on the correlative of Desargues' Theorem and on the properties of orthogonal involution pencils (Art. 58). In involution ranges there is no property strictly correlative to these properties of an involution pencil and consequently these theorems have no correlatives.

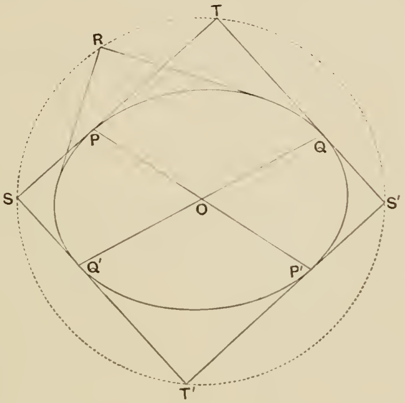
### Director Circle and Directrix.

*The locus of the points of intersection of tangents to a conic, which are at right angles, is for an ellipse or hyperbola a concentric circle (real or imaginary) termed the director circle, and for a parabola the directrix.*

*Ellipse and Hyperbola.*

Draw any two perpendicular tangents $TP$ and $TQ$ to touch the conic at $P$ and $Q$. Join $P$ and $Q$ to $O$, the centre of the conic, to meet the curve again in $P'$ and $Q'$, and let the tangents at $P'$ and $Q'$ intersect in $T'$. The four tangents at $P$, $Q$, $P'$, $Q'$ form a rectangle. Two of the pairs of opposite vertices are $T$ and $T'$, and $S$ and $S'$, and the third pair consists of the points at infinity in the directions $TQ$ and $TP$.

Take any other point $R$, the tangents from which to the conic are at right angles, and consider the involution determined at $R$ by the conic and the circumscribed quadrilateral $TS'T'S$.



Two pairs of conjugate rays of this involution are at right angles, viz. the pair of tangents from $R$ to the conic and the connectors of $R$ to the points at infinity on $TP$ and $TQ$. Therefore every pair of conjugate rays are at right angles. Therefore the angle $TRT'$ is a right angle, and the locus of $R$ is a circle described on $TT'$ as diameter. This circle has $O$ for its centre.

If the conic be an ellipse, and $A$, $A'$ and $B$, $B'$ be the ends of the major and minor axes, $Q$, $Q'$ and $P$, $P'$ may be taken to coincide with these points, and it follows that $OT^2 = OA^2 + OB^2$.

If the conic be a hyperbola and $OA$ and $OB$ be its semi-transverse and semi-conjugate axes, the tangents to the conic from points on the transverse axis at distances $\pm\sqrt{OA^2 - OB^2}$ are at right angles (Ex. (7), page 185). Hence in this case $OT^2 = OA^2 - OB^2$.

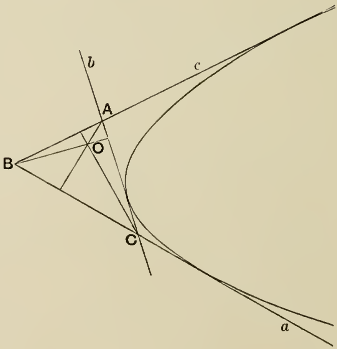
If the hyperbola is rectangular, the director circle becomes the centre of the hyperbola, and if the hyperbola is oblique the circle is imaginary.

*Parabola.* Let $T$ and $R$ be any two points the tangents from which to the parabola are at right angles, and let $TP$, $TQ$ be the tangents from $T$. Let $\infty$ represent the line at infinity. Consider the involution pencil determined at $R$ by the quadrilateral $TQ$, $TP$, $\infty$, $\infty$ and the parabola. The tangents from $R$ are at right angles as are also the lines joining $R$ to the points at infinity on $TP$ and $TQ$. Hence the involution pencil is orthogonal. Therefore $RT$ is perpendicular to the line joining $R$ to the point at infinity on the parabola, that is to its axis. Since $T$ may be taken on the axis, $RT$ is the directrix.

*Pair of Points.* If the conic become a pair of points (Art. 94) the director circle is the circle described on the line joining the points as diameter.

### (i) Properties of the Directrix of a Parabola.

(1) *The directrix of a parabola passes through the orthocentre of every circumscribed triangle.* or *The directrix of every parabola inscribed in a triangle passes through its orthocentre.*



Let $ABC$ be a triangle circumscribed to a parabola, $O$ its orthocentre, $a$, $b$, $c$ its sides and $\infty$ the line at infinity. Then $abc\infty$ is a circumscribed quadrilateral of the parabola. Consider the involution subtended by it at $O$. A pair of conjugate rays of this involution are the lines joining $O$ to $bc$ and $a\infty$. These are parallel to $OA$ and $a$ and are therefore at right angles. Similarly the lines joining $O$ to $ba$ and $c\infty$ are at right angles. Therefore the involution pencil is orthogonal and the tangents from $O$ to the parabola are at right angles. Therefore $O$ is on its directrix.

(2) *The directrix of a parabola inscribed in a quadrilateral passes through the orthocentre of each of the four triangles formed by the sides of the quadrilateral and therefore the four orthocentres are collinear.*

This follows from (1).

(3) *The circles described on the diagonals of a quadrilateral as diameters meet the line of orthocentres in the same pair of points and are therefore coaxal.*

Consider either of the points of intersection of two of the circles. The involution subtended at this point by the vertices of the quadrilateral is orthogonal. Hence the circle described on the third diagonal passes through the point and it is a point on the directrix of the inscribed parabola. Hence it is on the line of orthocentres of the quadrilateral.

From (3) it follows that the middle points of the diagonals of the quadrilateral, being the centres of a system of coaxal circles, are collinear, and that their line of collinearity is perpendicular to the line of orthocentres, which is the radical axis of the coaxal circles.

(ii) **Properties of the Director Circle of a Conic.**

(1) *The director circles of a system of conics inscribed in a quadrilateral are coaxal.*

(2) *The circles described on the diagonal of the quadrilateral as diameters are circles of the system.*

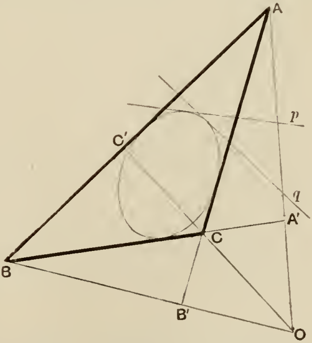
(3) *The radical axis is the directrix of the inscribed parabola, and is also the line of orthocentres of the quadrilateral.*

Consider one of the points of intersection of two of the director circles. The system of tangents to the conics from this point forms an orthogonal involution, and therefore the director circles of all the conics pass through the point, including the circles described on the diagonals as diameters and the directrix of the inscribed parabola. The same applies to the second point of intersection of the circles. Hence (1), (2) and (3) are true.

(4) *The self-polar circle of a triangle cuts the director circles of all conics inscribed in the triangle orthogonally.*

Let $ABC$ be the triangle, $a, b, c$ its sides, $O$ its orthocentre, and $AA'$, $BB'$, and $CC'$ the perpendiculars from the vertices on the opposite sides.

Then $O$ is the centre of the circle with regard to which the triangle $ABC$ is self-conjugate and $OB.OB'$ is equal to the square of its radius. (Example (5) Art. 88.)

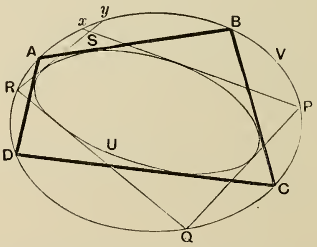


Draw any two straight lines $p$ and $q$. Describe a conic $S$ to touch $a$, $b$, $c$, $p$, $q$ and systems of conics to touch $a$, $b$, $c$, $p$ and $a$, $b$, $c$, $q$. The point $O$ is on the radical axis of the director circles of both these systems and therefore the tangents from $O$ to the director circles of both systems are equal. These must be equal to the tangents from $O$ to the director circle of $S$ and are therefore all equal.

The pair of points $A$, $B$ constitute a conic inscribed in $abc$ and the circle described on $AB$ is its director circle. This circle passes through $B'$. Hence the square of the tangent from $O$ to this circle equals $OB \cdot OB'$. Hence the squares of the tangents from $O$ to the director circles of all conics inscribed in $abc$ are equal to $OB \cdot OB'$. Therefore the circle with regard to which the triangle is self-conjugate cuts the director circles of all conics inscribed in $abc$ orthogonally.

*Corr.* This gives another proof of (i) (3), for, since the four self-conjugate circles of the triangles of a quadrilateral cut the director circles of the inscribed conics of the quadrilateral orthogonally, these latter are coaxal (Art. 88, Example (10)), and the orthocentres of the triangles are collinear and lie on the radical axis.

(*c*) *If one quadrangle can be inscribed in a conic, so that two pairs of opposite sides touch a second conic, an infinite number of quadrangles can be constructed in a similar manner.*



Let $ABCD$ be a quadrangle described about a conic $U$ and inscribed within a conic $V$. Let $PQRS$ be a second quadrangle described about $U$ so that $P$, $Q$, $R$ are on $V$ and let $PS$, $RS$ meet $V$ in $x$, $y$ respectively.

It is necessary to show that $S$, $x$ and $y$ coincide.

The lines joining $P$ (any point on $V$) to the vertices of the circumscribed quadrilateral $ABCD$ of the conic $U$ are pairs of conjugate rays of an involution of which the tangents from $P$ to the conic $U$ are conjugate rays (correlative of Desargues' Theorem). Therefore $(P \cdot AC, BD, Qx)$ is an involution pencil and $AC$, $BD$, $Qx$ form an involution on $V$. Similarly $(R \cdot AC, BD, Qy)$ is an involution pencil and $AC$, $BD$, $Qy$ form an involution on $V$.

Since these involutions on the conic $V$ are determined by $AC$, $BD$, and a point can have only one conjugate in a given involution, $x$ and $y$, which are conjugates of $Q$, must coincide and the quadrangle $PQRS$ is therefore inscribed in the conic $V$.

(*d*) *If two points are conjugate with regard to two conics of a system of conics through four fixed points, they are conjugates with regard to all conics of the system.*

The polars of a point $P$ with respect to the conics all pass through $P'$, the conjugate of $P$ with regard to the quadrangle formed by the four fixed points ((iii), Art. 106). Hence $P'$ must be the common conjugate of $P$ with respect to the two conics and is a common conjugate of $P$ with respect to all the conics of the system.

This may also be deduced from Desargues' Theorem.

(*e*) *The polars of any two fixed points with respect to a system of conics through four fixed points form two projective pencils.*

Let $P$ and $Q$ be any two fixed points, $l$ the line joining them, and $P_0, Q_0$ their common conjugates with respect to the system of conics ((*d*) above). Consider one of the conics which meets $l$ in real points $A$ and $A'$. The polars of $P$ and $Q$ will be two lines $p'$, $q'$ which pass through $P_0$ and $Q_0$ respectively and meet $l$ in a pair of points $P'$, $Q'$ which are harmonic conjugates of $P$ and $Q$ with respect to $A$, $A'$.

Consider a second conic of the system, which meets $l$ in $B$, $B'$;

*If two lines are conjugate with regard to two conics of a system of conics touching four fixed lines, they are conjugates with regard to all conics of the system.*

The poles of a line $p$ with respect to the conics all lie on a line $p'$, the conjugate of $p$ with respect to the quadrilateral formed by the four fixed lines (Corl. (iii), Art. 106). Hence $p'$ must be the common conjugate of $p$ with respect to the two conics and is a common conjugate of $p$ with respect to all the conics of the system.

This may also be deduced from the correlative of Desargues' Theorem.

*The poles of any two fixed lines with respect to a system of conics touching four fixed lines form two projective ranges.*

Let $p$ and $q$ be any two fixed lines, $L$ their point of intersection and $p_0$, $q_0$ their common conjugates with respect to the system of conics ((*d*) above). Consider one of the conics to which the tangents from $L$ are real lines $a$ and $a'$. The poles of $p$ and $q$ will be two points $P'$, $Q'$ which lie on $p_0$ and $q_0$ respectively and their connectors to $L$ are a pair of lines $p'$, $q'$ which are harmonic conjugates of $p$ and $q$ with respect to $a$, $a'$.

Consider a second conic of the system, the tangents to which from a second pair of polars $p''$, $q''$ of $P$ and $Q$ are obtained. These pass through $P_0$, $Q_0$ and meet $l$ in points $P''$, $Q''$, which are harmonic conjugates of $P$, $Q$ with respect to $BB'$.

Proceeding in a similar manner, on $l$ are obtained two ranges of points $P'P''P'''\ldots$ and $Q'Q''Q'''\ldots$ which are harmonic conjugates of $P$ and $Q$ with respect to $AA'$, $BB'$, $CC'$,… where the latter points, by Desargues' theorem, are pairs of conjugate points of an involution.

Hence (Example (7), Chapter VIII) the ranges $P'P''P'''\ldots$ and $Q'Q''Q'''\ldots$ are projective. Therefore the pencils $p'p''p'''\ldots$ and $q'q''q'''\ldots$ through $P_0$ and $Q_0$ are projective.

$L$ are $b$, $b'$; a second pair of poles $P''$, $Q''$ of $p$ and $q$ are obtained. These lie on $p_0$, $q_0$ and their connectors to $L$ are $p''$, $q''$, which are harmonic conjugates of $p$, $q$ with respect to $bb'$.

Proceeding in a similar manner, through $L$ are obtained two pencils of rays $p'p''p'''\ldots$ and $q'q''q'''\ldots$ which are harmonic conjugates of $p$ and $q$ with respect to $aa'$, $bb'$, $cc'$,… where the latter rays, by the correlative of Desargues' theorem, are pairs of conjugate rays of an involution.

Hence (correlative of Example (7), Chapter VIII) the pencils $p'p''p'''\ldots$ and $q'q''q'''\ldots$ are projective. Therefore the ranges $P'P''P'''\ldots$ and $Q'Q''Q'''\ldots$ on $p_0$ and $q_0$ are projective.

## EXAMPLES.

(1) Prove that an asymptote to a hyperbola is cut by the three pairs of opposite sides of a complete quadrangle inscribed in the hyperbola in three segments with a common middle point.

The double points of the involution determined on the asymptote are the points of contact of the two conics which can be described through the vertices of the quadrangle to touch the asymptote. One of these is the point at infinity at which the asymptote touches the hyperbola. Hence the other double point bisects the distances between all pairs of conjugate points.

(2) A circle has double contact with a hyperbola and from $P$ a point on the latter are drawn $PM$, parallel to an asymptote to meet the chord of contact in $M$, and $PT$ to touch the circle at $T$. Prove that $PM=PT$.

Let $PM$ meet the circle in $A$ and $A'$. Then $P$ is the centre and $M$ a double point of the involution determined by the hyperbola and circle on $PM$. Therefore $PM^2=PA\,.\,PA'=PT^2$.

(3) Given eight points in a plane and any straight line not passing through one of them, prove that in general two conics can be drawn, one passing through four points selected from the eight and one through the other four, so as to intersect each other in two points on the line.

The two points on the line are the common conjugates of the involutions determined on the line by the two quadrangles.

(4) Four points $A$, $B$, $C$, $D$ are taken on a conic. Any straight line through $D$ meets the conic again in $D'$ and the sides of the triangle in $A'$, $B'$, $C'$. Prove that $(A'B'C'D')$ is equal to $(ABCD)$.

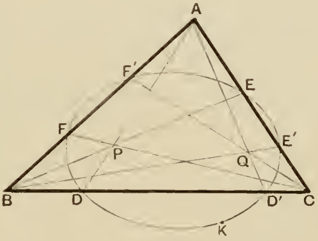
Let the tangent at $A$ meet $DD'$ in $A_1$. Then if $P$ be any point on the conic $(P.ABCD)=(A.ABCD)=(A_1C'B'D)$. But $(A_1C'B'D)=(A'B'C'D')$ by Desargues' theorem, for $AABC$ is an inscribed quadrangle of which two points coincide at $A$.

(5) If the four points of intersection of two parabolas are concyclic their axes are perpendicular.

The parabolas and the circle determine an involution on the line at infinity of which the points of contact of the parabolas are the double points. These points are therefore harmonic conjugates of the circular points at infinity. They therefore subtend a right angle at any point and the axes of the parabolas are consequently at right angles.

(6) Two parabolas whose axes are perpendicular intersect in four concyclic points.

(7) $ABC$ is a triangle and $D$, $E$, $F$ are three fixed points on $BC$, $CA$, $AB$, such that $AD$, $BE$, $CF$ are concurrent. A conic passing through $DEF$ and any fourth point meets $BC$, $CA$, $AB$ again in $D'$, $E'$, $F'$. Prove that $E'F'$, $F'D'$, $D'E'$ each pass through a fixed point, and that the point of concurrency of $AD'$, $BE'$, $CF'$ lies on a fixed conic circumscribing the triangle $ABC$.



Since the conic passes through four fixed points and the sides of the triangle each pass through one of these, the ranges $E'$ and $F'$ are in perspective (Art. 106 (i) ($a$)). $E'F'$ therefore passes through a fixed point. The pencils $BE'$, $CF'$ are projective. Therefore $Q$ describes a conic through $B$ and $C$ and by symmetry through $A$.

(8) Given two quadrangles in a plane prove that in general through any point $P$ in the plane may be drawn three straight lines such that on each the mates of $P$ in the two involutions determined on it by pairs of sides of the two quadrangles coincide.

Let $P$ be the given point and $ABCD$ and $A'B'C'D'$ the given quadrangles. Describe conics through $P$, $A$, $B$, $C$, $D$ and $P$, $A'$, $B'$, $C'$, $D'$. Let these conics intersect in $Q$, $R$, $S$. Consider the line $PQ$. $P$, $Q$ are conjugate points of the involution determined on $PQ$ by the sides of the first quadrangle: $P$, $Q$ are likewise conjugate points of the involution determined on $PQ$ by the sides of the second quadrangle. Therefore $PQ$ is one of the required lines. Similarly $PR$ and $PS$ are the other two required lines.

(9) Two conics intersect in four points $A$, $B$, $C$, $D$. Through $A$ and $B$ two chords $AEE'$, $BFF'$ are drawn to intersect the conics in $E$, $F$ and $E'$, $F'$ respectively. Then $EF$, $E'F'$ intersect on $CD$. See Art. 106 (i) ($a$).

(10) If the tangents to a conic $S$ at points $U$ and $V$ meet at $P$ and two lines be drawn through $P$ to meet the conic in $W$ and $X$, the conic $S'$ described through $U$, $V$, $W$, $X$ to touch $PW$ at $W$ will touch $PX$ at $X$.

Since $UVWX$ is a common inscribed quadrangle of the two conics, the tangents at $U$ and $V$ to $S$ and at $W$ and $X$ to $S'$ will intersect on the line joining $UX \cdot VW$ to $UW \cdot VX$, this line being the common polar of $UV \cdot WX$. Hence $P$ is on this line and, since the tangent at $W$ to $S'$ meets it at $P$, the tangent at $X$ to $S'$ likewise passes through $P$ and is the line $PX$.

(11) If two conics $S$ and $S'$ intersect in $U$, $V$, $W$, $X$ and $P$ the pole of $UV$ with respect to $S$ coincides with the pole of $WX$ with respect to $S'$, then the pole of $UV$ with respect to $S'$ coincides with the pole of $WX$ with respect to $S$.

Let the pole of $UV$ with respect to $S'$ be $R'$ and the pole of $WX$ with respect to $S$ be $R$. Then $P$, $R$, $R'$ are collinear being on the common polar of $UV \cdot WX$.

Let this line meet $S$, $S'$, and $UV$ and $WX$ in *$AA'$, $BB'$, $QQ'$. These points form an involution.

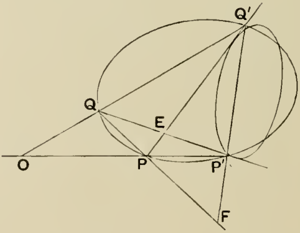
Since $P$ is the harmonic conjugate of $Q$ and $Q'$ with respect to $AA'$ and $BB'$, the range $AA'BB'$ is harmonic. (Example 9 (*b*), Art. 61.)

Hence $R$ and $R'$ the harmonic conjugates of $Q'$ and $Q$ with respect to $AA'$ and $BB'$ coincide. (Example 9 (*b*), Art. 61.)

(12) If from a fixed point $O$ a series of pairs of tangents be drawn to a system of conics which touch four given lines and these tangents meet a fixed conic of the system (to which $O$ is external) in variable points $P$, $P'$ and $Q$, $Q'$, show that the points of intersection of $PQ$, $P'Q'$ and of $PQ'$ and $P'Q$ are fixed points.

Let $PQ$, $P'Q'$ meet at $F$, and $PQ'$, $QP'$ at $E$. Then $EF$ is the polar of $O$ with respect to the fixed conic, and is a fixed line.

The tangents from $O$ to the conics form an involution pencil of which $OP$, $OQ$ and the pair of tangents from $O$ to the fixed conic are pairs of conjugate rays. But $OE$, $OF$ are common harmonic conjugates of these pairs of rays and are therefore the double rays of the involution. Therefore $E$ and $F$ are fixed.



(13) A quadrilateral has one pair of opposite angles each a right angle. Show that the inscribed parabola has the line joining these angular points for directrix and the point of intersection of the other diagonals for focus.

Since pairs of tangents to the parabola from points on the directrix intersect at right angles, the directrix is the line joining the vertices of the quadrilateral at which the right angles are situated. The focus is the pole of the directrix and therefore the opposite vertex of the diagonal triangle is the focus.

* This proof assumes that $A$, $A'$, $B$, $B'$ are real.

(14) Through a point $O$ straight lines $OP$, $OQ$ are drawn at right angles to one another to meet two given lines $AB$, $AC$—not passing through $O$—in $P$ and $Q$. Prove that the envelope of $PQ$ is a conic having $O$ for focus and touching $AB$, $AC$ in the points in which they are met by the straight line through $O$ perpendicular to $OA$.

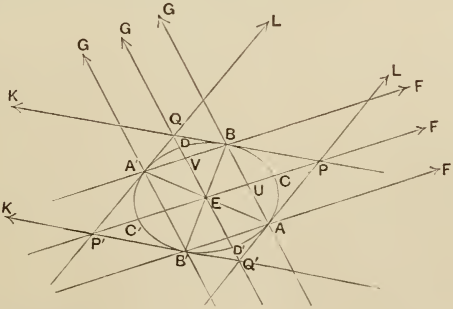
Since the pencils $OP$, $OQ$ are projective, the ranges $P$, $Q$ are projective and the envelope of $PQ$ is a conic touching $AB$, $AC$ at the points of the ranges corresponding to $A$, that is at the points where the line perpendicular to $OA$ meets these lines.

Let $OQ$, $OP$ meet $AB$, $AC$ in $P'$, $Q'$. Then $PP'Q'Q$ is a circumscribed quadrilateral of the conic. Therefore by the property of the complete quadrilateral circumscribed to the conic, $P'Q$ and $PQ'$ are conjugate lines. But as these are any pair of conjugate lines through $O$ and they are at right angles, $O$ is a focus of the conic.

(15) $A$, $B$, $C$, $D$ are any four points on a given conic and the two conics, which pass through $A$, $B$, $C$, $D$ and touch a directrix of the given conic, are drawn. Show that the portion of the directrix, intercepted between the points of contact, subtends a right angle at the corresponding focus.

If $P$ and $Q$ are the points at which the two conics touch the directrix, they are conjugate points with regard to these conics and therefore with regard to all conics through $A$, $B$, $C$, $D$, Art. 106 ($d$). Hence $P$ and $Q$ are conjugate points with regard to the given conic. Since $S$ the focus is the pole of the directrix, $SP$ and $SQ$ are a pair of conjugate lines, and, being conjugate lines through a focus, are at right angles.

## 107. Complete inscribed quadrangle and circumscribed quadrilateral, and conics in self-perspective.



**Particular Cases.**

If the ends $AA'$, $BB'$ of two diameters of a conic be taken as the vertices of an inscribed quadrangle, and the circumscribed quadrangle be constructed, then

(1) One of the vertices of the diagonal points triangle ($E$) becomes the centre of the conic and the other two ($G$ and $F$) are at infinity.

Let the finite point of intersection of the tangents at $A$, $A'$, $B$, $B'$ be $P$, $P'$, $Q$, $Q'$.

Let $EP$ and $EQ$ meet $BA$ and $BA'$ in $U$ and $V$.

(2) $EU$ and $EV$ are conjugate diameters of the conic, for, since $BAUG$ is harmonic and $G$ is at infinity, each bisects chords parallel to the other.

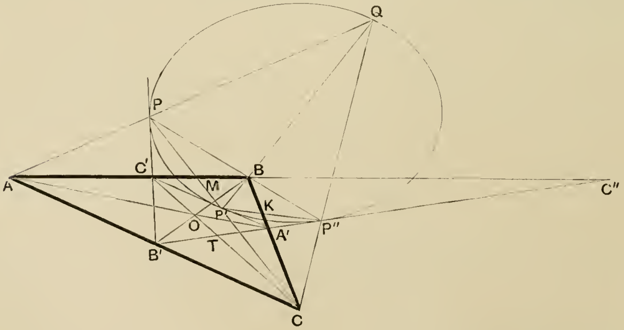
(3) The supplemental chords $AB$ and $A'B$ are parallel to $EV$ and $EU$ and are therefore parallel to conjugate diameters.

(4) Since the lines $EP$ and $EQ$ are conjugate diameters, any tangent meets a pair of parallel tangents in points whose connectors to the centre are a pair of conjugate diameters.

Certain theorems in connexion with the conic can be conveniently proved from the fact that a conic is in self-perspective, if the perspective is harmonic, and any vertex and the opposite side of a self-conjugate triangle are the centre and axis of perspective.

**Deductions.**

(a) *If $ABC$ be a triangle self-conjugate with regard to a conic and any tangent to the conic meets the sides $AB$ and $AC$ in $C'$ and $B'$ and the line connecting $A$ to the point $CC'$. $BB'$ meets the side $BC$ in $A'$, then $A'C'$ and $A'B'$ are tangents to the conic.*



Let the point $BB'$ . $CC'$ be $O$ and let $A'B'$ meet $AB$ in $C''$ and $CC'$ in $T$. Then from the quadrangle $ABA'B'$ the range ($B'A'TC''$) and consequently the pencil ($C'$ . $B'A'CB$) are harmonic. Therefore in the harmonic perspective, centre $C$ and axis $AB$, $C'B'$ and $C'A'$ are corresponding lines. But in this perspective the conic corresponds to itself and therefore, since $C'B'$ is a tangent, $C'A'$ is also a tangent. Similarly $B'A'$ touches the conic.

From the preceding the following theorems are obtained :

(*b*) *If a triangle be inscribed in a conic in such a way that its sides pass through the vertices of another triangle, which is self-conjugate with respect to the conic, the two triangles are in perspective.*

In the preceding the points of contact of $B'C'$, $C'A'$, $A'B'$ form such an inscribed triangle. $Q$ is the centre of perspective.

(*c*) *If two triangles are inscribed in, and in perspective with, a given triangle, a conic can be described to touch their six sides and to have the given triangle for a self-conjugate triangle.*

Describe a conic, which has the given triangle for a self-conjugate triangle, to touch one side of each of the other triangles. This conic will touch their six sides.

(*d*) *If a triangle be circumscribed to a conic so that its vertices lie on the sides of a triangle which is self-conjugate with regard to the conic the triangles are in perspective.* This is the converse of (*a*).

(*e*) *If two triangles are circumscribed to and in perspective with a given triangle, a conic can be described to pass through their six vertices and to have the given triangle for a self-conjugate triangle.*

Describe a conic which has the given triangle for a self-conjugate triangle and passes through one of the vertices of each of the other triangles. This conic will pass through their six vertices.

(*f*) *If a triangle* $ABC$ *be self-conjugate with regard to a conic and a triangle* $A'B'C'$ *be circumscribed to the given triangle so as to be in perspective with it, then if one vertex of the latter triangle is on the conic the other two are likewise on the conic.*

Let $A'$ be on the conic, then the vertices $B'$ and $C'$ may be constructed as harmonic perspectives of $A'$ with respect to the vertices and sides of the triangle.

## EXAMPLES.

(1) Two triangles are inscribed in a given triangle and are in perspective with it. Show that a conic can be circumscribed to the first triangle with respect to which the other two triangles are self-conjugate.

The required conic is the conic through the vertices of the triangle and the two centres of perspective.

(2) A parabola is drawn touching the sides $AB$, $BC$, $CD$, $DA$ of the cyclic quadrilateral $ABCD$. Show that its directrix passes through the intersection of $AC$ and $BD$.

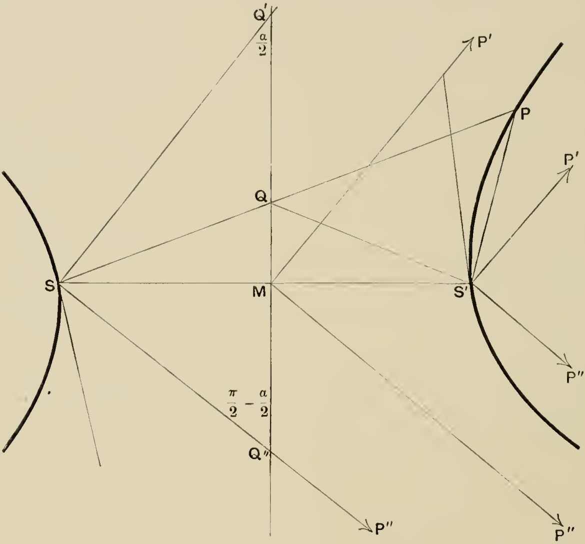
Let $AB.CD$ and $BC.DA$ be $E$ and $F$ respectively. Then since $ABCD$ is inscribed in a circle the circle round $BEC$ and $CFD$ will intersect in the focus of the parabola and (Example (7), Chapter XI) on the line $EF$. But $AC.BD$ is a vertex of a self-conjugate triangle of the parabola of which $EF$ is the opposite side ; therefore, as the focus is on $EF$, the directrix passes through $AC.BD$.

(3) Prove that a conic may be described to touch the eight sides of two quadrilaterals which have the same diagonal triangle.

(4) $EFG$ is a triangle self-conjugate with regard to a conic. Prove that, if through $E$ any two lines, harmonic conjugates of $EF$ and $EG$, be drawn to meet the conic in $K$, $L$, $M$, $N$, then $EFG$ is the diagonal points triangle of the quadrangle $KLMN$.

## Rectangular Hyperbola.

**108.** *A rectangular hyperbola is the locus of the points of intersection of corresponding rays of two oppositely equal pencils.* (Art. 94 (iv).)



*Construction.*

Let $S$ and $S'$ be any two fixed points and $M$ the middle point of $SS'$. Through $M$ draw a line $MQ$ perpendicular to $SS'$. On $MQ$ take any point $Q$. Join $SQ$, $S'Q$ and through $S'$ draw a line $S'P$ making any given angle $a$ with $S'Q$. Then $SQ$ and $S'P$ are two corresponding rays of oppositely equal pencils and (by definition) $P$ their point of intersection for different positions of $Q$ describes a rectangular hyperbola.

*Asymptotes.*

If a ray $SQ'$ be drawn to make an angle $\frac{a}{2}$ with $MQ$ it will be parallel to the corresponding ray of the other pencil. It therefore determines a point at infinity on the curve and is parallel to an asymptote.

If $SQ''$ be drawn perpendicular to $SQ'$ it will likewise be parallel to its corresponding ray and therefore $SQ''$ is the direction of the other asymptote. Hence the asymptotes are at right angles. This is the property from which the hyperbola is named.

To the ray $SS'$ of the pencil, centre $S$, corresponds a ray through $S'$ making an angle $a$ with $S'S$. Hence the tangent to the locus at $S'$ makes an angle $a$ with $S'S$. The tangent at $S$ to the locus likewise makes an angle $a$ with $SS'$. Hence the tangents at $SS'$ are parallel and therefore $SS'$ is a diameter and $M$ is the centre of the rectangular hyperbola.

*Construction of the tangent.*

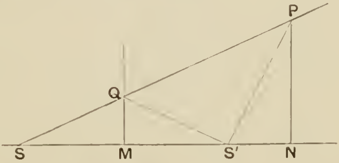
Since the asymptote $MP'$ makes an angle $\frac{\pi}{2}-\frac{a}{2}$ with $SS'$ and the tangent at $S'$ makes an angle $a$ with $SS'$, both the tangent at $S'$ and $MS'$ make equal angles $\frac{\pi}{2}-\frac{a}{2}$ with the asymptote. Hence, to construct the tangent at any point $P$ of the curve, join $P$ to the centre $M$ and draw through $P$ a line making the same angle with the asymptote, that $PM$ makes with it.

It follows that pairs of conjugate diameters of a rectangular hyperbola are equally inclined to the asymptotes.

*Axes.*

If the angle $a$ is taken to be $\frac{\pi}{2}$ the tangents at $S$ and $S'$ are both perpendicular to $SS'$, and $SS'$ is the transverse axis of the rectangular hyperbola.

*If from any point $P$ on a rectangular hyperbola a perpendicular $PN$ is drawn to the axis and $S$ and $S'$ are the ends of the axis, then $PN^2 = SN \, . \, S'N$.*



In the figure $QS'P$ is a right angle, and therefore the angle

$$PSN = QS'M = S'PN.$$

Therefore the triangles $PSN$ and $S'PN$ are similar, and

$$\frac{PN}{SN} = \frac{S'N}{PN}.$$

Therefore $PN^2 = SN \, . \, S'N$,

also since $SN = MN + MS'$ and $S'N = MN - MS'$,

$$MN^2 - PN^2 = S'M^2.$$

By considering the special case when $a$ is zero, it is seen that the pair of lines $MQ$ and $SS'$, which are at right angles, form a rectangular hyperbola. Hence any pair of lines at right angles may be looked upon as a rectangular hyperbola.

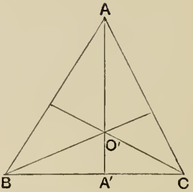
*If two conics described through four points are rectangular hyperbolas, all conics through these points are rectangular hyperbolas.*

The conics determine an involution on the line at infinity. Since two of the conics are rectangular hyperbolas, two pairs of the conjugate points on this line are determined by orthogonal rays. Hence all pairs of conjugate points are determined by orthogonal rays. Therefore all conics through the four points have their asymptotes at right angles, and are rectangular hyperbolas.

*All conics through the vertices of a triangle and its orthocentre are rectangular hyperbolas.*

Any side and the perpendicular from the opposite vertex constitute a rectangular hyperbola. Therefore there are two rectangular hyperbolas through the four points, and all conics through these points are rectangular hyperbolas.

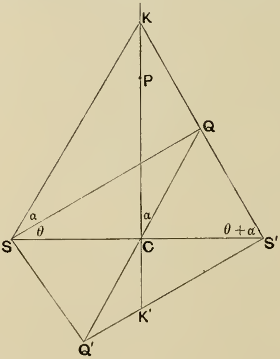
*If a rectangular hyperbola passes through the vertices of a triangle it also passes through the orthocentre.*



Let $ABC$ be the triangle and let the given rectangular hyperbola meet $AA'$ the perpendicular to $BC$ in $O'$. Then through $A$, $B$, $C$, $O'$ there are two rectangular hyperbolas, viz. the given one and that made up of the lines $AO'$, $BC$. Therefore every conic through these points is a rectangular hyperbola. Therefore $BO'$ and $AC$ which constitute a conic through these points are a rectangular hyperbola and are at right angles. Therefore $O$ is the orthocentre of the triangle.

*A system of rectangular hyperbolas, described with any two points $S$ and $S'$ for the ends of a diameter, determines the same involution on the perpendicular diameter.*

Let $C$ be the middle point of $SS'$. Then, if the generating pencil be rotated through an angle $\alpha$ the direction of one of the asymptotes is inclined at an angle $\frac{\alpha}{2}$ to $CP$, the perpendicular diameter through $C$. Hence the diameter $CQ$ conjugate to $CP$ is inclined at an angle $\frac{\pi}{2}-\alpha$ to $CS'$.



Let $Q$ and $Q'$ be the points in which this diameter meets the rectangular hyperbola. Let $S'Q$ and $S'Q'$ meet $CP$ in $K$ and $K'$. Then, since $QQ'$ passes through the pole of $CP$, $K$ and $K'$ are a pair of conjugate points of the involution determined on $CP$ by the rectangular hyperbola.

In the figure if $CSQ$ be $\theta$, then $CS'Q$ is $\theta+\alpha$. The angle $KSC=$angle $KS'C$. Hence angle $KSQ=\alpha$. Hence a circle passes through $K$, $Q$, $C$, $S$. Therefore the angle $SQK=$angle $SCK=$a right angle.

Hence $QS'Q'$ is a right angle and the circle on $KK'$ as diameter passes through $S$ and $S'$.

Therefore $\quad CK \,.\, CK' = -CS^2.$

The involution determined by $K$ and $K'$ of which $C$ is the centre is therefore the same whatever $\alpha$ may be and is therefore the same for all rectangular hyperbolas of the system.

## EXAMPLES.

(1) If the normal at $P$ to a rectangular hyperbola meets the conic again in $P'$ then $PP'$ subtends a right angle at the other extremity of the diameter through $P$.

(2) $A$ and $B$ are the ends of a diameter of a rectangular hyperbola and $CD$ the perpendicular diameter. $P$ is any point on the rectangular hyperbola. Prove that, if $AP$ meets $CD$ in $Q$, the angle $PBQ$ is constant.

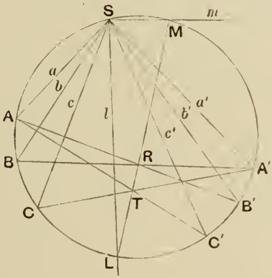
This follows from the construction of a rectangular hyperbola.

# CHAPTER XVII

## CONSTRUCTION OF SELF-CORRESPONDING ELEMENTS. PROBLEMS OF SECOND DEGREE. CONSTRUCTION OF CONICS UNDER VARIOUS CONDITIONS

**109. To construct the self-corresponding elements—when real—of two superposed projective forms.**

*To construct the self-corresponding elements of two superposed projective pencils* ($S.abc\ldots$) *and* ($S.a'b'c'\ldots$).

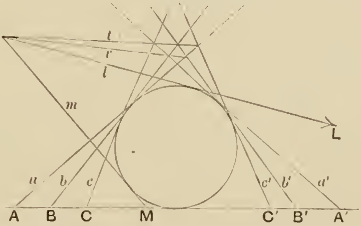


(i) Through $S$ describe any circle or conic to meet three pairs of corresponding rays of the pencils in $A$, $B$, $C$ and $A'$, $B'$, $C'$.

Let $R$ and $T$ be respectively the points $AB'.A'B$ and $AC'.A'C$.

Let $RT$ meet the circle or conic in $M$ and $L$. Then since $M$ and $L$ are the self-corresponding points of the ranges $ABC\ldots$ and $A'B'C'\ldots$ on the conic (Art. 105 ($b$)),

*To construct the self-corresponding elements of two superposed projective ranges* ($s.ABC\ldots$) *and* ($s.A'B'C'\ldots$).



Describe any circle or conic to touch $s$ and let $a$, $b$, $c$ and $a'$, $b'$, $c'$ be the tangents to this curve from three pairs of corresponding points of the ranges.

Let $r$ and $t$ be respectively the lines $ab'.a'b$ and $ac'.a'c$.

Let the tangents from $rt$ to the circle or conic be $m$ and $l$. Then since $m$ and $l$ are the self-corresponding tangents of the projective systems of tangents $abc\ldots$ and

$SM$ and $SL$ are the self-corresponding rays of the pencils.

(ii) Take any circle and let any tangent $s$ to the circle cut the pencils in $A$, $B$, $C$ and $A'$, $B'$, $C'$. Construct by the right-hand side of (i) the self-corresponding points $L$ and $M$ of these ranges.

The rays joining these points to $S$ will be the self-corresponding rays of the pencils, viz. $l$ and $m$.

$a'b'c'\ldots$ to the conic (Art. 105 ($b$)), $sm$ and $sl$ are the self-corresponding points of the ranges.

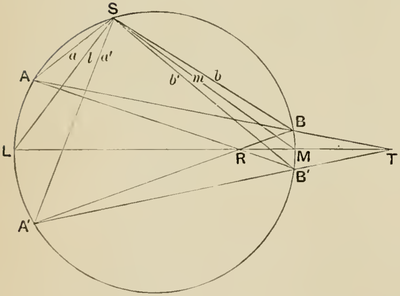
Take any circle and project from any point $S$ on the circle the ranges into pencils $(S\,.\,abc\ldots)$ and $(S\,.\,a'b'c'\ldots)$. Construct by the left-hand of (i) the self-corresponding rays $l$ and $m$ of these pencils.

These rays will meet $s$ in the self-corresponding points of the ranges, viz. $L$ and $M$.

**110. To construct the double elements—when real—of an involution form.**

*To construct the double elements of an involution pencil*

$(S.aa'bb'\ldots)$.



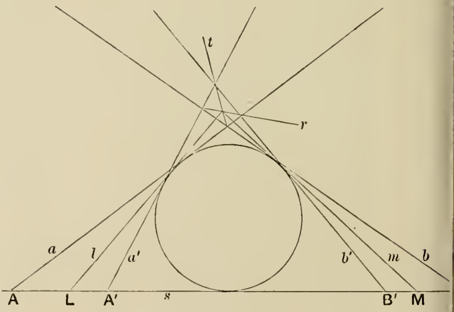
(i) Through $S$ describe any circle or conic to meet two pairs of corresponding rays in $A$, $A'$, $B$, $B'$.

Let $R$ and $T$ be the points $AB'\,.\,A'B$ and $AB\,.\,A'B'$.

Let the line $RT$ meet the circle or conic in $L$ and $M$.

*To construct the double elements of an involution range*

$(s\,.\,AA'BB'\ldots)$.



Describe any circle or conic to touch $s$ and let the tangents from two pairs of corresponding points of the ranges be $a$, $a'$, $b$, $b'$.

Let $r$ and $t$ be the lines $ab'\,.\,a'b$ and $ab\,.\,a'b'$.

Let the tangents from $rt$ to the circle or conic be $l$ and $m$.

Then since $L$ and $M$ are the double points of the involution on the circle or conic (Art. 75), $SL$ and $SM$ are the double rays of the involution pencil.

(ii) Take any circle and let $s$ any tangent to this circle cut the involution in $AA'BB'$.

Construct by the right-hand side of (i) the double elements $L$ and $M$ of this involution range.

The rays joining these points to $S$ will be the double rays of the involution pencil.

Then since $l$ and $m$ are the double rays of the involution system of tangents to the circle or conic (Art. 75), $sl$ and $sm$ are the double points of the involution.

Take any circle and project from $S$ any point on the circle the range into a pencil $(S . aa'bb')$.

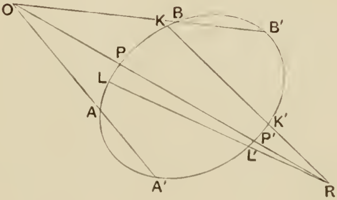
Construct by the left-hand side of (i) the double elements $l$ and $m$ of this involution pencil.

The points in which these rays meet $s$ will be the double points of the involution range.

**111. To find the common pair of conjugates—when real—of two involutions situated on a conic** (cf. Art. 55).

Let the involutions be determined by the pairs of conjugate points $AA'$, $BB'$ and $KK'$, $LL'$. Let $AA' . BB'$ be $O$ and $LL' . KK'$ be $R$. Join $OR$.

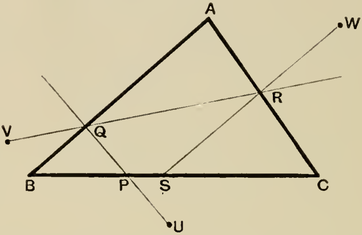
Then the points $P$, $P'$, if real, in which $OR$ meets the conic are the common pair of conjugates of the two involutions. (Art. 95 (*c*).)



By the correlative method the two tangents, which are a pair of conjugates in two involutions of tangents to the conic, may be constructed.

**112. Method of false positions.** In attempting to solve a geometrical problem it frequently happens that the solution of the problem is found to depend on finding the configuration for which two points coincide. Such points are frequently situated on some given line or conic and in this case it often happens that for different positions of one of the points it may be proved that the second point describes a range projective with that described by the first. In such cases the problem can immediately be solved. Any three positions of the first point may be taken and the three corresponding positions of the second. The projective ranges are then completely determined and their self-corresponding points can (by Arts. 109 and 110) be at once constructed. These self-corresponding points give the required configuration. The following are instances of the application of this method:

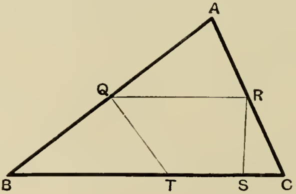
(*a*) *To inscribe a triangle in a given triangle so that its sides may pass through three given points.*



Let $ABC$ be the given triangle and $U$, $V$, $W$ the given points. Through $U$ draw a transversal to meet $BC$ and $BA$ in $P$ and $Q$. Join $V$ to $Q$ to meet $AC$ in $R$. Join $W$ to $R$ to meet $BC$ in $S$.

Then the range described by $P$ is projective with the range described by $Q$. This is projective with the range described by $R$, which is projective with the range described by $S$. Therefore $P$ and $S$ describe two superposed projective ranges, and the self-corresponding points of these give the solution of the problem.

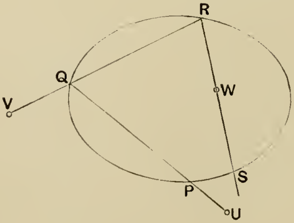
(*b*) *To inscribe a square in a given triangle in such a way that one of its sides may lie along a side of the triangle.*



Take any point $Q$ on the side $BA$ of the triangle. Through $Q$ draw a parallel to $BC$ to meet $AC$ in $R$. Through $R$ draw a perpendicular to $QR$ to meet $BC$ in $S$. Draw $QT$ making an angle of 45° with $QR$.

Then, for different positions of $Q$, the range described by $Q$ is projective with the range described by $R$ and therefore with the range described by $S$. It is also projective with the range described by $T$. Hence the ranges described by $T$ and $S$ are projective. The self-corresponding points of these ranges each give the position of a vertex of the square.

(*c*) *To inscribe a triangle in a given conic so that its sides may pass through three given points.*



Let $U$, $V$, $W$ be the given points. Through $U$ draw a chord to meet the conic in $P$ and $Q$. Join $Q$ to $V$ to meet the conic again in $R$. Join $R$ to $W$ to meet the conic in $S$.

Then the range described by $P$ is projective with the range described by $Q$. This is projective with the range described by $R$, which is projective with the range described by $S$.

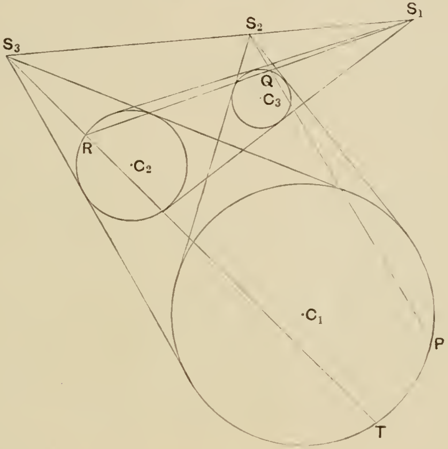
Hence the ranges described by $P$ and $S$ are projective and their self-corresponding points give possible vertices of the required triangle.

The correlative construction gives the solution of the following problem:

*To circumscribe a triangle to a given conic so that its vertices shall lie on three given lines.*

The above and likewise the correlative solution may be extended to the case in which the triangle is replaced by a polygon, whose sides pass through fixed points or whose vertices lie on fixed lines.

(*d*) *To describe a circle to touch three given circles.*



It has been proved

(1) that the six centres of similitude of three circles lie three by three on four straight lines termed the axes of similitude. (Ex. 18, Chapter XI.)

(2) that if a circle touch two given circles its points of contact are collinear with a centre of similitude of the circles. (Ex. 15, Chapter XI.)

Construct $S_1$, $S_2$, $S_3$ three collinear centres of similitude of the three given circles. Take $P$ any point on the circle centre $C_1$. John $P$ to $S_2$ to meet circle centre $C_3$ at $Q$. Join $Q$ to $S_1$ to meet circle centre $C_2$ at $R$. Join $R$ to $S_3$ to meet circle centre $C_1$ at $T$.

Then (Art. 84 (11)) for different positions of $P$ the ranges $T$, $R$, $Q$, $P$ on the circles are projective. Determine the self-corresponding points of the ranges $T$ and $P$. Let $P'$, $Q'$, $R'$ be the positions of $P$, $Q$, $R$, when $P'$ is one of these points.

Since the triangles $P'Q'R'$ and $C_1C_2C_3$ are in perspective $P'C_1$, $Q'C_3$, $R'C_2$ meet at a point $O$. This point is the centre of the circles, which by the converse of (2) touch the given circles in pairs at $T''$, $Q'$, $R'$. It is therefore the centre of a circle which touches the three given circles.

(e) *Given two straight lines $l$, $l'$ and two parabolas $S$, $S'$, find $P$ on $l$ and $P'$ on $l'$ such that the tangents from $P$ to $S$ are parallel to the tangents from $P'$ to $S'$.*

The pairs of tangents from the points on $l$ to $S$ determine conjugate points of an involution on the line at infinity. (Art. 75.)

Similarly the pairs of tangents from points on $l'$ to $S'$ determine conjugate points of an involution on the line at infinity.

The pair of common conjugates of these two involutions determine the points, where the required tangents meet the line at infinity, *i.e.* they determine the directions of the required tangents.

### 113. Construction of a Conic.

In Art. 94 it was shown how conics of various kinds can be constructed by means of the anharmonic properties of the conic. Conics may, however, be constructed by means of any of the other three fundamental theorems or by means of their correlatives. The methods are as follows:

*Construction by means of points.*

(A) To construct a conic through five given points.

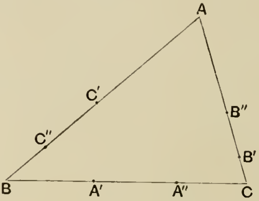
(i) *By Carnot's Theorem.*

Let $A'$, $A''$, $B'$, $B''$, $C'$ be the five points. Through $C'$ draw any straight line meeting $A'A''$ and $B'B''$ in $B$ and $A$ respectively, and let $C$ be the point of intersection of $A'A''$ and $B'B''$.

Then $C''$, the point in which the conic meets $AB$, is determined uniquely by the relation

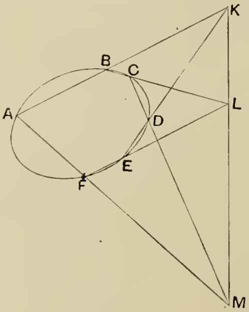
$$\frac{BA'}{CA'}\cdot\frac{BA''}{CA''}\cdot\frac{CB'}{AB'}\cdot\frac{CB''}{AB''}\cdot\frac{AC'}{BC'}=\frac{BC''}{AC''}.$$

Hence any number of points on the conic may be found.



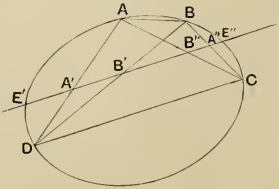
(ii) *By Pascal's Theorem.*

Let $A$, $B$, $C$, $D$, $E$ be the given points. Through $E$ draw any straight line $EL$. Let $AB$ and $ED$ intersect in $K$, and $BC$ and $EL$ in $L$. Join $KL$ to meet $DC$ in $M$. Then $EL$ and $AM$ intersect in a point $F$ on the conic.



(iii) *By Desargues' Theorem.*

Let $A$, $B$, $C$, $D$, $E'$ be the given points. Through $E'$ draw any transversal meeting $AD$ and $BC$ in $A'$, $A''$ and $BD$ and $AC$ in $B'$, $B''$. Then in the involution determined by the two pairs of conjugates $A'A''$, $B'B''$ construct $E''$ the conjugate of $E'$ (Art. 56). Then $E''$ is the point in which the transversal is met by the conic.



The above are unique solutions.

(B) *To construct a conic to pass through three given points and to touch a given line at a given point.*

This is a particular case of (A). The necessary modifications are as follows:

(i) Let $A'$ and $A''$ coincide; $BC$ becomes the given line which is touched by the conic at $A'$.

(ii) Let $B$ and $C$ coincide; $BCL$ becomes the given line which is touched by the conic at $B$.

(iii) Let $B$ and $C$ coincide; $BC$ becomes the given line which is touched by the conic at $B$.

The above are unique solutions.

(C) *To construct a conic to pass through one given point and to touch two given lines at given points.*

This is likewise a special case of (A). To obtain the construction the following modifications are necessary in (B).

(i) Let $B'$ also coincide with $B''$.

(ii) Let $D$ also coincide with $E$.

(iii) Let $B$ and $D$ coincide and likewise $A$ and $C$. Then $B'$ and $B''$ are a pair of conjugate points of the involution and $AB$ meets the transversal in a double point. Hence $E''$ the conjugate of $E'$ can be found.

The above are unique solutions.

(D) *To construct a conic to pass through four given points and to touch a given line.*

Let $A$, $B$, $C$, $D$ be the given points. They determine on the given line an involution. Let $E$ and $F$ be the double points of this involution. A conic through $A$, $B$, $C$, $D$ and $E$ or $F$ will by Desargues' theorem touch the given line at $E$ or $F$.

In this case there are two solutions.

(E) *To construct a conic to pass through three given points and to touch two given lines.*

Let the given points be $Q$, $Q'$, $Q''$ and the given lines $p$, $p'$. Consider a conic touching $p$ and $p'$ and passing through $Q$ and $Q'$. It is seen (Art. 106 (ii)) that the chord of contact of this conic with $p$ and $p'$ passes through one or other of two fixed points on $QQ'$. Similarly, by considering a conic touching $p$ and $p'$ and passing through $Q$ and $Q''$, it is seen that the chord of contact passes through one or other of two fixed points on $QQ''$. Therefore the chord of contact is one or other of four fixed lines.

In this case there are four solutions.

(F) *To construct a conic to pass through two given points, to touch a given line at a given point and to touch a second line.*

Let the given points be $B$, $C$; $A$ the point at which the given line $a$ must be touched and $l$ the second given line.

The points at which the required conic will touch $l$ are the double points of the involution determined by $A$, $A$, $B$, $C$ on $l$.

In this case there are two solutions.

*Constructions by means of lines.*

By using the correlative methods the following problems may be solved.

(A) *To construct a conic to touch five given lines.* (One solution.)

(B) *To construct a conic to touch three given lines and a given line at a given point.* (One solution.)

(C) *To construct a conic to touch a given line and to touch two given lines at given points.* (One solution.)

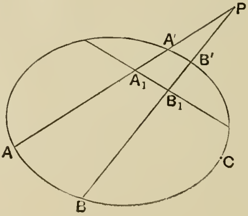
(D) *To construct a conic to touch four given lines and pass through a given point.* (Two solutions.)

(E) *To construct a conic to touch three given lines and pass through two given points.* (Four solutions.)

(F) *To construct a conic to touch two given lines, a given line at a given point and to pass through a second point.* (Two solutions.)

### 114. Constructions involving pole and polar, and conjugate points.

($a$) *To construct a conic such that a given point may be the pole of a given line with respect to it and such that it* (1) *passes through three given points,* (2) *passes through two given points and touches a given line,* (3) *passes through one given point and touches two given lines, or* (4) *touches three given lines.*



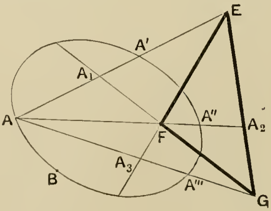
(1) Let $P$ be the given point, $p$ its polar, and $A$, $B$, $C$ the given points on the conic. Join $P$ to $A$ and $B$ to meet $p$ in $A_1$ and $B_1$. Take $A'$ and $B'$ the harmonic conjugates of $A$ and $B$ with regard to $PA_1$ and $PB_1$. The conic through $ABCA'B'$ is the required conic.

(2) If the conic touches a line $c$ instead of passing through $C$. Proceed as in (1) but describe a conic through $ABA'B'$ to touch $c$.

(3) and (4) are the correlatives respectively of (2) and (1).

(1) and (4) have one solution while (2) and (3) have two solutions.

($b$) *To construct a conic such that a given triangle may be self-conjugate with respect to it and such that it* (1) *passes through two given points,* (2) *passes through a given point and touches a given line, or* (3) *touches two given lines.*



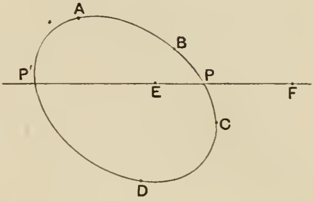
(1) Let $EFG$ be the given triangle and $A$ and $B$ the given points. Join $A$ to $E$, $F$, $G$ to meet the opposite sides in $A_1$, $A_2$, $A_3$; take $A'$, $A''$, $A'''$ the harmonic conjugates of $A$ with respect to $A_1E$, $A_2F$, $A_3G$. The conic through $A$, $A'$, $A''$, $A'''$ and $B$ is the required conic.

(2) Proceed as in (1) but describe a conic through $A$, $A'$, $A''$, $A'''$ to touch the given line $b$.

(3) is the correlative of (1).

(1) and (3) have one solution while (2) has two.

(*c*) *To construct a conic through four given points such that a pair of given points may be conjugate with regard to it**.



Let $A$, $B$, $C$, $D$ be the given points and $E$ and $F$ the pair of conjugate points.

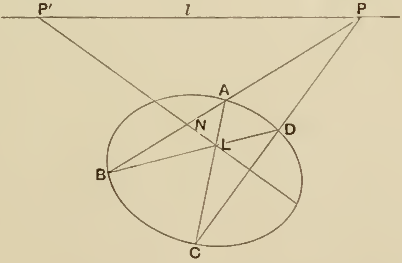
$A$, $B$, $C$, $D$ determine on $EF$ an involution of which $P$ and $P'$, the points where $EF$ meets the conic, are a pair of conjugate points. If $E$, $F$ are conjugate points with respect to the conic, $P$ and $P'$ are harmonic conjugates of $E$ and $F$. Therefore $P$ and $P'$ are a pair of conjugate points of the involution of which $E$ and $F$ are double points. $P$ and $P'$ are therefore determined as the common pair of conjugates of two involutions. They can be found by Art. 55. Hence six points on the conic are given.

A similar construction holds when the conic instead of passing through four points satisfies other conditions.

The correlative theorem is:

*To construct a conic to touch four given straight lines such that a given pair of lines are conjugate with regard to it.*

(*d*) *To describe a conic through three given real points and through the two double points (real or imaginary) of a given involution**.



Let $l$ be the line on which the given involution is situated and let $A$, $B$, $C$ be the three given points. Let $AB$ meet $l$ in $P$ and let $P'$ be the point of the involution conjugate to $P$. On $AB$ take $N$ the harmonic conjugate of $P$ with respect to $AB$. Join $P'N$ to meet $AC$ in $L$. Let $BL$ and $PC$ meet in $D$. Then since $P'N$ is the polar of $P$, $D$ is a point on the curve. Similarly other points $D_1$ and $D_2$ may be determined by considering the points where $AC$ and $BC$ meet $l$. The conic through $A$, $B$, $C$, $D$, $D_1$, $D_2$ is the required conic.

The correlative theorem is:

*To describe a conic to touch three real lines and to touch the two double rays (real or imaginary) of an involution pencil.*

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* See also Chapter XXII.

(*e*) *Given a conic determined either by five points or by five tangents to construct:*

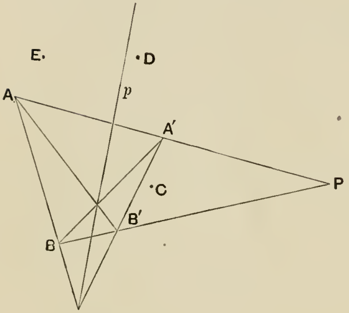
(1) *The polar of a given point,*

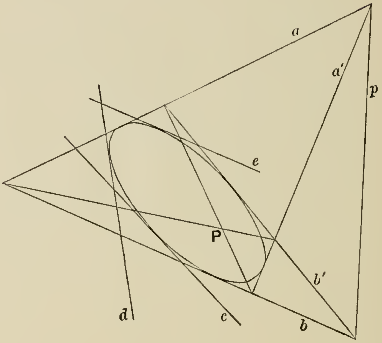
(2) *The pole of a given line,*

(3) *The points of intersection of a given line with the conic,*

(4) *The tangents from a given point to the conic.*

If five points on a conic are given, the five tangents at these points may be constructed by Pascal's theorem, and if five tangents are given their points of contact can be constructed by Brianchon's theorem. Hence the conic may be regarded as determined either by five points or by five tangents.





(1) Let the conic be determined by $A, B, C, D, E$ and let $P$ be the given point. Join $A$ and $B$ to $P$ and find by Pascal's theorem $A'$ and $B'$ the other points in which these lines meet the conic. Then $p$ the line joining $AB' . BA'$ to $AB . A'B'$ is the polar of $P$.

cf. p 215

(3) Let the conic be determined by $A, B, C, D, E$ and let $p$ be the given line. Take any pair of points $P$ and $Q$ on the line $p$ and determine their polars. These will meet $p$ in a pair of points $P'$ and $Q'$ which are conjugates of $P$ and $Q$ with respect to the conic. Determine on $p$ the double points of the involution $PP'$, $QQ'$. These double points are the points of intersection of the line with the conic.

(2) Let the conic be determined by $a, b, c, d, e$ and let $p$ be the given line. Through the points where $a$ and $b$ meet $p$ draw by Brianchon's theorem the other tangents $a'$ and $b'$ to the conic. Then $P$ the point of intersection of $ab' . ba'$ and $ab . a'b'$ is the pole of $p$.

(4) Let the conic be determined by $a, b, c, d, e$ and let $P$ be the given point. Take any pair of lines $p$ and $q$ through the point $P$ and determine their poles. The lines $p'$ and $q'$ joining these points to $P$ are conjugates of $p$ and $q$ with respect to the conic. Determine in the pencil vertex $P$ the double rays of the involution $pp'$, $qq'$. These double rays are the pairs of tangents from $P$ to the conic.

(*f*) *To describe a conic through* 4, 3, 2, 1 *or* 0 *given points, so that* 1, 2, 3, 4 *or* 5 *pairs of given points may be conjugates with regard to it.*

Let the given points be $A$, $B$, $C$, $D$, and let the connectors of the pairs of conjugate points be $a$, $b$, $c$, $d$, $e$. The conic in each case has to pass through points on these lines, which are harmonic conjugates of the pairs of conjugate points, that is through pairs of conjugate points of involutions on these lines. Such involutions will be called the involutions $a$, $b$, $c$, $d$ or $e$.

(1) Draw two conics through $A$, $B$, $C$ and through two pairs of conjugate points of involution $e$. These meet at a point $P$. By Art. 101, converse, the conic through $A$, $B$, $C$, $D$ and $P$ is the required conic.

(2) Draw two conics through $A$, $B$, $C$ and two pairs of conjugate points of involution $e$, and two other conics through $A$, $B$, $C$ and two pairs of conjugate points of involution $d$. These pairs of conics intersect in two points $P$ and $P'$. The conic $A$, $B$, $C$, $P$, $P'$ is the required conic.

(3) Construct as in (2), a conic $S$ through $A$, $B$, $C$, $P$, $P'$ and a second conic $S'$ substituting some points $R'$ for $C$. These conics intersect in four points $A$, $B$, $C'$, $D'$. Any conic through these points passes through conjugate points of the involutions $d$ and $e$. Determine from $A$, $B$, $C'$ and involution $c$ a point $Q$ as in (1). Then $ABC'D'Q$ is the required conic.

(4) Determine from $A$, $B$, and the involutions $c$, $d$, $e$ a conic $S$ as in (3). Substitute a point $R''$ for $B$ and determine a second conic $S'$. The conics $S$ and $S'$ intersect in four points $A$, $B''$, $C''$, $D''$. Any conic through these points meets $c$, $d$, $e$ in conjugate points of the given involutions. Determine $T$ as in (1) from $A$, $B''$, $C''$ and the involution $b$. Then $A$, $B''$, $C''$, $D'$, $T$ is the required conic.

(5) Determine for $A$ and the involutions $b$, $c$, $d$, $e$ a conic $S$ as in (4). From any point $R'''$ and the involutions $b$, $c$, $d$, $e$ determine similarly a conic $S'$. The conics $S$ and $S'$ intersect in four points $A'''$, $B'''$, $C'''$, $D'''$, and any conic through these points passes through conjugate points of involutions $b$, $c$, $d$, $e$. Determine from $A'''$, $B'''$, $C'''$ and the involution $a$ a point $E'''$. Then the conic $A'''B'''C'''D'''E'''$ passes through conjugate points of the given involutions on $a$, $b$, $c$, $d$, $e$.

### *Particular Cases.*

**115.** Particular cases of the preceding frequently arise, but with a little ingenuity they may be solved by the methods already given, if it is borne in mind that the data in the left-hand column below are only particular cases of those in the right-hand column.

| | |
|---|---|
| The direction of an asymptote. | A point at infinity on the curve. |
| An asymptote. | A point on the curve and the tangent at the point. |
| The centre. | The pole of a given line. |
| An axis. | The polar of a given point, which point is at infinity on any perpendicular line. |
| Two axes. | A self-conjugate triangle. |
| A focus. | Two tangents which are double rays of an involution. |
| Two foci. | Four tangents. |
| A diameter. | A pair of conjugate lines. |
| A pair of conjugate diameters. | A self-conjugate triangle. |

If the conic is :

(1) a parabola, a tangent is given ;

(2) a rectangular hyperbola, a pair of conjugate points with respect to the conic are given, viz. the circular points at infinity ;

(3) a circle, a pair of points on the conic, which are the double points of an involution, are given.

The following are special constructions for the parabola and hyperbola.

*Parabola.*

(1) Given two tangents $a$ and $b$ to a parabola and their points of contact $A$ and $B$ to construct the curve.

Let $O$ be the middle point of $AB$. Join $ab$ to $O$ and let the point at infinity on this line be $\infty$. Through $ab$ draw any line to meet $A\infty$ and $B\infty$ in $D$ and $E$ respectively. Then $AE.DB$ ($\equiv F$) is a point on the curve.

This follows from the Pascal hexagon $AAFBB\infty$.

(2) Given four tangents to construct a parabola.

Let two of the tangents $t$ and $t'$ meet the other two tangents $a$ and $b$ in $A$ and $A'$ and in $B$ and $B'$. On $a$ and $b$ construct two similar ranges in which $A$ and $A'$ and $B$ and $B'$ are pairs of corresponding points.

Then the connectors of the corresponding points envelope the parabola (Art. 102, Ex. (1)).

*Hyperbola.*

Given three points $A$, $B$, $C$ on the curve and the directions of the asymptotes to construct the curve.

Through $A$ and $C$ draw lines parallel to the asymptotes to meet in $K$. Through $K$ draw any line to meet $AB$ and $CB$ in $R$ and $S$. Parallels through $R$ and $S$ to the asymptotes meet in a point $P$ on the curve.

By Pascal's theorem.

## EXAMPLES.

(1) Given five points on a conic determine by a linear construction the pole of the line joining two of them. Art. 105 (i).

(2) Construct a parabola to pass through four given points and show that two parabolas can be drawn in general to pass through four points.

(3) Prove that two equilateral hyperbolas, if any, can be drawn touching four given lines.

(4) Construct a hyperbola given three points on the curve and an asymptote.

(5) Show how to construct a conic given a focus and three points on it ; and prove that there are four such conics.

(6) Given five points on a conic and any parallelogram, find a construction for the centre by means of ruler only.

(7) Given five points on a hyperbola construct its asymptotes.

(8) Given three tangents of a conic and one of the foci, construct the second focus and any number of tangents.

# CHAPTER XVIII

## TRIANGLE LOCI AND ENVELOPES. POLE AND POLAR LOCI AND ENVELOPES. ANHARMONIC AND GENERAL LOCI AND ENVELOPES

**116.** In this chapter certain problems and theorems concerning loci and envelopes are considered. Those at the commencement of Art. **117** are in themselves of considerable importance and should be carefully studied.

### Triangle Loci and Envelopes.

The anharmonic property of a conic and its correlative theorem together with the converse of these theorems give a solution of many problems concerning loci, in which the loci are conic sections. The following are instances. It is assumed in speaking of a triangle that $A$, $B$, $C$ are the vertices opposite respectively to the sides $a$, $b$, $c$.

1. If the three sides of a triangle $a$, $b$, $c$ pass respectively through fixed points $P$, $Q$, $R$, and two vertices $A$ and $B$ move on fixed lines $d$ and $e$, then the third vertex $C$ describes a conic through $P$ and $Q$. (Cf. Ex. 1, Art. 71.)

The pencil described by $b$, which passes through $Q$ is projective with that described by $a$, which passes through $P$; for these pencils are projective with the ranges described by $A$ and $B$ on $d$ and $e$, which are themselves projective.

2. If the sides $a$ and $b$ of a triangle pass through the fixed points $P$ and $Q$ and the vertices $A$ and $B$ move along fixed lines $d$ and $e$ and the side $c$ touches a conic which also touches $d$ and $e$, then the vertex $C$ describes a conic, which passes through $P$ and $Q$.

The proof is identical with the preceding, except that the ranges described

If the three vertices of a triangle $A$, $B$, $C$ lie respectively on three fixed lines $p$, $q$, $r$, and two sides $a$ and $b$ pass through fixed points $D$ and $E$, then the third side $c$ envelopes a conic touching $p$ and $q$. (Cf. Ex. 1, Art. 71.)

The range described by $B$ on $q$ is projective with that described by $A$ on $p$; for these ranges are projective with the pencils described by $a$ and $b$ through $D$ and $E$, which are themselves projective.

If the vertices $A$ and $B$ of a triangle lie on fixed lines $p$ and $q$ and the sides $a$ and $b$ pass through fixed points $D$ and $E$ and the vertex $C$ lies on a conic which also passes through $D$ and $E$, then the side $c$ envelopes a conic, which touches $p$ and $q$.

The proof is identical with the preceding, except that the pencils described by $A$ and $B$ on $d$ and $e$ are projective because $c$ is a tangent to a conic which touches $d$ and $e$.

by $a$ and $b$ through $D$ and $E$ are projective because $C$ is on a conic through $D$ and $E$.

3. Through two fixed points $A$ and $B$ lines are drawn to a variable point $P$ on a fixed straight line. If lines $AC$, $BC$ are drawn through $A$ and $B$ such that the angles $PAC$, $PBC$ have given values, the locus of $C$ is a conic through $A$ and $B$.

The pencils described by $AC$ and $BC$, through $A$ and $B$, are projective.

Two fixed straight lines $a$ and $b$ are intersected at $A$ and $B$ by a variable line $p$, which passes through a fixed point. If a line $c$ meet $a$ and $b$ in points $A'$ and $B'$ such that the lengths $AA'$ and $BB'$ have given values, the envelope of $c$ is a conic touching $a$ and $b$.

The ranges described by $A'$ and $B'$ on $a$ and $b$ are projective.

4. If in the preceding the lines $AC$ and $BC$ cut off equal or proportional intercepts on given lines, the locus of $C$ is a conic through $A$ and $B$.

If in the preceding the lengths $AA'$ and $BB'$ subtend equal angles at two given points, the envelope of $C$ is a conic touching $a$ and $b$.

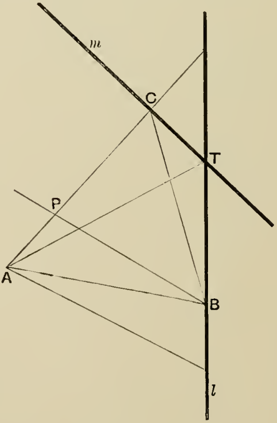
5. In a triangle given in species* if one vertex be fixed and another move along a straight line,

(1) the envelope of the side opposite the fixed vertex is a parabola, of which the fixed vertex is a focus;

(2) the locus of the point of intersection of a side which passes through the fixed vertex with a line, drawn in a fixed direction through the opposite vertex, is a conic.

It should be noticed that the sides of triangles given in species determine three projective ranges on the line at infinity.

(1) Let the vertex $A$ be fixed and let $B$ move along the straight line $l$. Then $CB$ determines on the line at infinity a range projective with that determined by $AB$, and therefore with the range described by $B$ on $l$. Hence the envelope of $CB$ is a parabola touching the line $l$.



Similarly the parabola touches the line $m$ on which (Addendum 6 ($a$)) $C$ moves. If $m$ and $l$ meet at $T$ the points $C$, $T$, $B$, $A$ are concyclic and therefore $A$ is the focus of the parabola. Example (3), Chapter XIII.

(2) If a line through $B$ drawn in a fixed direction meets $AC$ at $P$, $P$ is the point of intersection of rays of two projective pencils, whose vertices are at infinity in the given direction and at $A$. Hence the locus is a conic through $A$.

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* In Addendum 6 some important theorems connected with triangles, given in species, are proved.

6. Find the envelope of a side of a triangle given in species and inscribed in a given triangle.

If the triangle $A'B'C'$ be inscribed in the triangle $ABC$, the circumcircles of $AC'B'$, $BC'A'$ and $CA'B'$ meet at a fixed point $O$, and the triangle $OA'B'$ is given in species (Addendum 6 (*b*)). Since $O$ is fixed and $A'$ and $B'$ lie on two fixed lines the envelope of $A'B'$ is a parabola of which $O$ is the focus.

7. If a series of similar figures are described, three corresponding points of which are situated on the sides of a triangle, then the loci of all other points of the figures are straight lines, and the envelopes of all lines are parabolas which have the same point for focus.

In the last example let $P'$ be a variable point such that $A'B'C'P'$ is always similar to a given figure. Then the triangle $OA'P'$ is given in species, $O$ is fixed and $A'$ moves along a fixed line. Hence the locus of $P$ is a fixed line.

If $P'$, $Q'$ be any two points of the figure, $OP'Q'$ forms a triangle given in species of which one vertex $O$ is fixed while the other vertices $P'$, $Q'$ move along straight lines. Therefore the envelope of $P'Q'$ is a parabola of which $O$ is the focus.

### 117. Pole and Polar Loci and Envelopes.

1. *Given four points on a conic the polar of a fixed point passes through a fixed point.*

This is equivalent to proving that,

*Every point has one common conjugate point with respect to a system of conics through four fixed points.*

Let $A$, $B$, $C$, $D$ be the four fixed points through which the system of conics is described and $S$ the point, the envelope of whose polars with respect to the conics is required. Construct $S'$ the conjugate of $S$ with respect to the quadrangle $ABCD$. Then the polar of $S$ with respect to every conic through $A$, $B$, $C$, $D$ passes through $S'$. Art. 106 (iii).

If a conic be described through the points $A$, $B$, $C$, $D$, $S$, the tangent to this conic at $S$ will

*Given four tangents to a conic the pole of a fixed line lies on another fixed line.*

This is equivalent to proving that,

*Every line has one common conjugate line with respect to a system of conics touching four fixed lines.*

Let $a$, $b$, $c$, $d$ be the four fixed lines which are touched by the conics of the system and $s$ the line, the locus of whose poles with respect to the conics is required. Construct $s'$ the conjugate of $s$ with respect to the quadrilateral *abcd*. Then the pole of $s$ with respect to every conic touching $a$, $b$, $c$, $d$ lies on $s$. Correlative of Art. 106 (iii).

If a conic be described to touch the lines $a$, $b$, $c$, $d$, $s$ and $S$ be the point of contact of this cut the conics of the system in an involution of which $S$ is a double point (by Desargues' theorem). Let $S''$ be the other double point of the involution.

Then $S$ and $S''$ are harmonic conjugates of every pair of conjugates of the involution determined on $SS'$ and are therefore conjugate points with respect to all the conics of the system, which meet $SS''$ in real points. Hence $S''$ is the same point as $S'$.

conic with $s$, the pairs of tangents from $S$ to the conics of the system will form an involution of which $s$ is a double ray (by Correlative of Desargues' theorem). Let $s''$ be the other double ray of the involution.

Then $s$ and $s''$ are harmonic conjugates of every pair of tangents from $S$ to the conics of the systems and are therefore conjugate lines with respect to all conics of the system, the tangents to which from $ss''$ are real. Hence $s''$ is the same line as $s'$.

## 2. **The Eleven points locus and its correlative.**

*Given a system of conics through four fixed points* (*a*) *the locus of the common conjugates of points on a fixed line and* (*b*) *the locus of the poles of the fixed line with respect to the conics of the system, is a conic which passes through* (i) *the six harmonic conjugates of the points where the fixed line meets each side of the quadrangle of fixed points with respect to the vertices on that side,* (ii) *the three vertices of the common self-conjugate triangle of the conics and* (iii) *the two points at which conics of the system touch the given line.*

Let $P$ be any point on the fixed line $l$ and let $L$ and $L'$ be the poles of $l$ with respect to two given conics (1) and (2) of the

*Given a system of conics touching four fixed lines* (*a*) *the envelope of the common conjugate lines of lines through a fixed point and* (*b*) *the envelope of the polars of the fixed point with respect to the conics of the system, is a conic which touches* (i) *the six harmonic conjugates of the lines joining the fixed point to the vertices of the quadrilateral with respect to the pairs of sides of the quadrilateral which meet at that vertex,* (ii) *the three sides of the common self-conjugate triangle of the conics and* (iii) *the two lines which are tangents to conics of the system at the fixed point.*

Let $p$ be any line through the fixed point $L$ and let $l$ and $l'$ be the polars of $L$ with respect to two given conics (1) and (2) of the system. The polars of $P$ with respect to the conics (1) and (2) will be two straight lines $LQ$ and $L'Q$ which pass through $L$ and $L'$ and meet at $Q$ a point which is a common conjugate of $P$ with respect to all the conics of the system. (Art. 117, 1.)

As $P$ moves along the line $l$, $LQ$ and $L'Q$ describe two pencils through $L$ and $L'$ which are each equi-anharmonic with the range described by $P$ and are therefore projective with each other.

Hence the locus of $Q$ is a conic through $L$ and $L'$, which is therefore the locus of common conjugates of points on $l$ with respect to conics through the four fixed points.

Since $L$ and $L'$ are the poles of $l$ with respect to the conics (1) and (2), which are any two conics of the system, the poles of $l$ with respect to every conic of the system lie on this conic, which is therefore the locus of poles of $l$ with respect to all the conics of the system.

(i) If the four points through which the conics of the system pass are $A$, $B$, $C$, $D$ and $l$ meet $AB$ in $R$, and $R'$ be the harmonic conjugate of $R$ with respect to $A$ and $B$, then $R'$ is a common conjugate of $R$. Therefore the locus passes through $R'$ and the five other points similarly constructed.

system. The poles of $p$ with respect to the conics (1) and (2) will be two points $lq$ and $l'q$ which lie on $l$ and $l'$ and have for their connector $q$ which is a common conjugate of $p$ with respect to all the conics of the system.

(Art. 117, 1.)

As $p$ rotates round $L$, $lq$ and $l'q$ describe two ranges along $l$ and $l'$ which are each equi-anharmonic with the pencil described by $p$ and are therefore projective with each other.

Hence the envelope of $q$ is a conic touching $l$ and $l'$, which is therefore the envelope of common conjugates of lines through $L$ with respect to conics touching the four fixed lines.

Since $l$ and $l'$ are the polars of $L$ with respect to the conics (1) and (2), which are any two conics of the system, the polars of $L$ with respect to every conic of the system are tangents to this conic, which is therefore the envelope of polars of $L$ with respect to all the conics of the system.

If the four lines which are touched by the conics of the system are $a$, $b$, $c$, $d$ and the connector of $L$ to $ab$ be $r$, and $r'$ be the harmonic conjugate of $r$ with respect to $a$ and $b$, then $r'$ is a common conjugate of $r$. Therefore the envelope touches $r'$ and the five other lines similarly constructed.

(ii) If $EFG$ be the diagonal points triangle of $ABCD$ and $l$ meet $EF$ in $T$, then $G$ will be a common conjugate of $T$, since the polars of $T$ with respect to all conics of the system pass through $G$. Hence the locus passes through $G$, and similarly through $E$ and $F$.

If $efg$ be the diagonal triangle of the quadrilateral $abcd$ and the connector of $L$ to $ef$ be $t$, then $g$ will be a common conjugate of $t$, since the poles of $t$ with respect to all conics of the system lie on $g$. Hence the envelope touches the line $g$, and similarly it touches $e$ and $f$.

(iii) The locus passes through the points where the two conics of the system touch the line $l$, for these points are poles of the line $l$ with respect to conics of the system. These points may be constructed as the double points of the involution determined by the quadrangle $ABCD$ on $l$.

The envelope touches the two lines which are tangents at $L$ to conics of the system, for the lines are polars of the point $L$ with respect to conics of the system. These lines may be constructed as the double rays of the involution determined by the quadrilateral $abcd$ at $L$.

3. *Transversals are drawn across a quadrangle in such a way that the locus of one of the double points of the involutions, in which they are cut, is a straight line. Prove that the locus of the other double point is a conic circumscribing the harmonic triangle of the quadrangle.*

The double points of the involution in which the transversal is cut are harmonic conjugates of its intersections with conics described through the vertices of the quadrangle and are therefore common conjugate points of the system of conics. Consequently, since one describes a straight line, the other describes the conic obtained in 2.

4. *Prove that a conic, which circumscribes the harmonic triangle of a quadrangle and which passes through the double points of the involution determined by the quadrangle on a given transversal, passes also through the six points which are the harmonic conjugates with respect to every pair of vertices of the quadrangle, of the points where the transversal meets the side of the quadrangle containing that pair.*

The conic is the locus obtained in 2.

5. *The locus of the centres of a system of conics circumscribed to a quadrangle is a conic circumscribing its diagonal points triangle and passing through the six middle points of the sides.*

(See Example 10, Art. 103.)

*The locus of the centres of a system of conics inscribed in a quadrilateral is a straight line passing through the middle points of the diagonals.*

The centre of a conic is the pole of the line at infinity, which is a fixed line, consequently the above is a particular case of Art. 117, 2.

The centre of a conic is the pole of the line at infinity, which is a fixed line, consequently the above is a particular case of Art. 117, 1.

The theorem on the right-hand side may also be proved as follows:

Let $a$, $b$, $c$, $d$ be the four lines touched by the conics of the system. Find the direction of the point at infinity on the parabola which touches $a$, $b$, $c$, $d$ (*i.e.* the point of contact with the line at infinity of the conic touching $abcd$ and $\infty$). The line through the centre of any conic of the system parallel to this direction is the locus of the centres. For, by the correlative of Desargues' theorem, if $I$ be the point at infinity on the parabola, this line and the line at infinity are the double rays of the involution formed by drawing pairs of parallel tangents from $I$ to the conics and the middle points of the chords of contact of such tangents are the centres of the conics and lie on the line in question.

Since the limiting forms of the conics are the three pairs of opposite vertices, the locus passes through the middle points of the diagonals.

6. *The locus of the centres of a system of conics which pass through the vertices of a triangle and through its orthocentre is the nine-points circle of the triangle.*

The system of conics is a system of conics through four fixed points and therefore this theorem is a particular case of 5. As two of the pairs of lines which determine the involution on the line at infinity are at right angles the double points of this involution, through which the locus passes, are the circular points at infinity. The locus of centres is therefore a circle. This locus passes through the middle points of the connectors of the four points and is therefore the nine-points circle. It also passes through the feet of the perpendiculars which are the diagonal points of the quadrangle.

The conics of the system are rectangular hyperbolas. (Art. 108.)

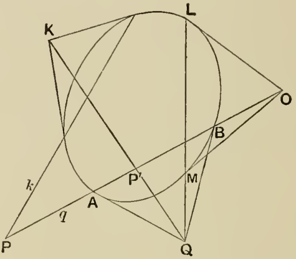
7. The locus of the centres of hyperbolas, which pass through three given points and have an asymptote given in direction, is a hyperbola, which has an asymptote in the given direction.

This is a particular case of the locus of the centre of a system of conics through four points, one point being at infinity. The middle points of the lines joining the point at infinity to the three given points are at infinity in the direction of the asymptote and therefore an asymptote of the locus is in that direction.

8. Prove that the locus of the centre of a conic which passes through the centres of the inscribed and escribed circles of a triangle is the circumscribing circle.

If $A'$, $B'$, $C'$ be the centres of the escribed circles and $O$ the centre of the inscribed circle, $O$ is the orthocentre of the triangle $A'B'C'$, and by 6 the locus of the centres of conics through $A'B'C'$ and $O$ is the nine-points circle of $A'B'C'$, *i.e.* the circumcircle of $ABC$.

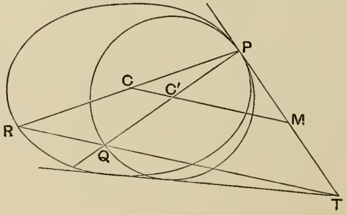
9. The connector of a pair of conjugate points with respect to a given conic passes through a fixed point and one of the pair lies on a given straight line; show that the locus of the other is a conic passing through (1) the fixed point, (2) the intersections of the given conic with the given straight line and with the polar of the fixed point, and (3) the pole of the given straight line.



Let $O$ be the point through which the variable chord $q$ passes, meeting the conic in $A$ and $B$, and the fixed line $k$ (whose pole is $K$) in $P$. Let $Q$ be the pole of $q$ which is on $LM$ the polar of $O$.

Then $QK$ is the polar of $P$ and the point $P'$, where it meets the line $q$, is the conjugate of $P$ with regard to the conic. But the pencil $q$ is projective with the range described by its pole $Q$ on $LM$. Therefore the pencils $q$ and $KQ$ are projective and the locus of $P'$ is a conic through $K$ and $O$. It obviously passes through $L$ and $M$. At the points where $k$ meets the conic $P'$ coincides with $P$ and therefore these points are also on the locus.

10. A circle and a conic osculate at $P$ and have diameters $PQ$, $PR$. Show that the line $QR$, the tangent at $P$ and the other common tangent are concurrent.

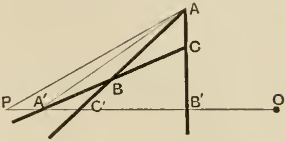


Let $C$ and $C'$ be the centres of the conic and circle. Let the tangent at $P$ meet the other common tangent at $T$ and $CC'$ at $M$. Then from the nature of the locus of centres of conics touching four straight lines, $M$ bisects $PT$. Therefore $QR$, which is parallel to $CC'M$, passes through $T$.

## 118. Loci and Envelopes by Anharmonic Ratios.

1. *A variable line passes through a fixed point: to find the locus of a point on the line which forms with the three points in which the line meets the sides of a given triangle a range of a constant anharmonic ratio.*

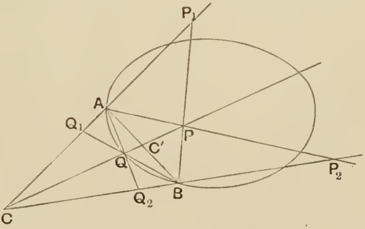
Through $O$ the fixed point draw any transversal to meet the sides of the triangle in $A', B', C'$. Let $P$ be a point such that $(A'B'C'P)$ is constant. Join $A'$ and $P$ to $A$. Then the pencil $(A\,.\,A'B'C'P)$ has the given constant anharmonic ratio. Therefore since $AC'$, $AB'$ are fixed the pencils described by $AP$ and $AA'$ are projective. But, since $A'$ is situated on the fixed line $BC$, the pencils $AA'$ and $OA'$ are projective. Hence the pencils $AP$ and $OA'$ are projective. Therefore the locus of $P$ is a conic through $A$ and $O$. By symmetry this conic passes through $B$ and $C$.



Hence, if a system of conics be described through four points, and through one of the points a transversal be drawn to meet one of the conics in $P$ and the sides of the triangle formed by the other three fixed points in $A'B'C'$, then the anharmonic ratio of the range $(A'B'C'P)$ is constant for every transversal. Also by taking a different conic of the system the value of this anharmonic ratio may be made equal to any given quantity.

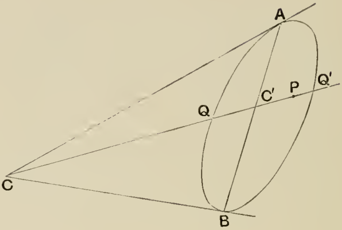
2. *If a variable transversal be drawn through one vertex $C$ of a triangle $ABC$ to meet the opposite side in $C'$ and a conic through $A$, $B$ in $Q$, find the locus of a point $P$ on this transversal such that $(CC'QP)$ is constant.*

Project $Q$ and $P$ from $B$ on to $AC$ into the points $Q_1$, $P_1$ respectively, and from $A$ on to $CB$ into the points $Q_2$, $P_2$. Then the pencils $AQ$, $BQ$ are projective and therefore the ranges $Q_1$, $Q_2$ are projective. But, since the anharmonic ratios of the ranges $CAQ_1P_1$ and $CBQ_2P_2$ are constant and $C$, $A$ and $B$ are fixed, the range $Q_1$ is projective with the range $P_1$ and the range $Q_2$ with the range $P_2$. Hence the ranges $P_1$ and $P_2$ are projective. Therefore the pencils $AP$ and $BP$ are projective and the locus of $P$ is a conic through $A$ and $B$.



3. *If a variable chord $CQQ'$ of a conic be drawn through a fixed point $C$ and a point $P$ be taken in this chord such that $(CQPQ')$ is constant, then the locus of $P$ is a conic.*

Let $CA$, $CB$ be the tangents from $C$ to the conic and let the chord of contact meet $AB$ in $C'$.



Then

$$(CQPQ') = K \text{ (a constant).}$$

But $\quad (CQC'Q') = 2.$

Therefore

$$\frac{(CQPQ')}{(CQC'Q')} = (CQPC') = \frac{K}{2}.$$

Hence by the last example $P$ describes a conic through $A$ and $B$.

4. *A conic touches two sides $CA$ and $CB$ of a triangle $ABC$. Any tangent is drawn to the conic meeting $BC$ in $P$, $AC$ in $P'$ and $AB$ in $Q$. On the tangent a point $N$ is taken such that $PP'QN$ is harmonic. Find the locus of $N$.*

Let $CN$ meet $AB$ in $K$ and let $AP \, . \, BP'$ be $L$. The ranges described by $P$ and $P'$ on $CB$ and $CA$ are projective, therefore the pencils $AP$, $BP'$ are projective and the locus of $L$ is a conic through $A$ and $B$. From the harmonic property of the quadrangle $ABPP'$, the pencil $C \, . \, ABLQ$ is harmonic and therefore $L$ lies on $CN$. Hence, since $(CLKN)$ is harmonic and $L$ describes a conic through $A$ and $B$, the locus of $N$ is also a conic through $A$ and $B$.

5. If a variable triangle $ABC$ is such that two of its vertices $A$, $B$ lie respectively on two fixed tangents to a given conic, while $AB$ always touches the conic, and the sides $BC$, $CA$ pass respectively through two fixed points $P$, $Q$, then $C$ will lie on a fixed conic which passes through $P$ and $Q$.

6. Fixed straight lines $OA$, $OB$ are cut in $P$, $Q$ by a line which passes through a fixed point. Prove that the centre of the circumcircle of the triangle $OPQ$ lies on a hyperbola.

Take $M$ and $N$ the middle point of $OP$, $OQ$ and erect perpendiculars at $M$ and $N$ to meet at $R$. $R$ is the centre of the circumcircle of $OPQ$.

Ranges $P$ and $Q$ are projective, therefore the ranges $M$ and $N$ are projective. Hence the pencils formed by joining $M$ and $N$ to the points at infinity in directions perpendicular to $OA$ and $OB$ are projective. Therefore the locus of $R$ is a conic through the points at infinity perpendicular to $OA$ and $OB$, *i.e.* it is a hyperbola.

7. If $S_1$ and $S_2$ be two points which project three collinear points $A_1$, $B_1$, $C_1$ and three collinear points $A_2$, $B_2$, $C_2$ respectively into the same three collinear points, prove that $S_1S_2$ touches a conic.

Let $S_1S_2$ meet the two bases in $D_1$, $D_2$. Then if $D_1$ is given $D_2$ is uniquely determined for $(A_1B_1C_1D_1) = (A_2B_2C_2D_2)$. Therefore for different positions of $S_1$ and $S_2$, the ranges $D_1$ and $D_2$ are projective. Therefore $D_1D_2$ (or $S_1S_2$) touches a conic which also touches the two bases.

8. A perpendicular is drawn to any tangent to a parabola at the point where it meets a fixed tangent: prove that the envelope of this line is a parabola which touches the fixed tangent at the point where it is met by the directrix of the given parabola.

Let $a$ be the fixed tangent and $v$ the line at infinity. Let the variable tangent meet the fixed tangent and the line at infinity in $P$ and $P'$. If a perpendicular to the tangent be drawn through $P$, it will meet the line at infinity in a point $Q'$ which is the conjugate of $P'$ in a definite involution. Therefore the ranges $P$, $Q'$ are projective and $PQ'$ will envelope a conic which touches the fixed tangent and the line at infinity, *i.e.* a parabola. The point where it touches $a$ will be the point on $a$ corresponding to $av$. At this point the variable tangent must be perpendicular to $a$ and therefore this point is a point on the directrix.

9. Prove that the locus of the point in which the perpendicular from a fixed point to a variable tangent to a parabola meets the corresponding diameter is a rectangular hyperbola passing through the fixed point, and find its asymptotes.

Let the perpendicular from the fixed point $O$ meet the tangent at $P$ in $N$ and the corresponding diameter in $Q$. The perpendicular from the focus $S$ will meet the diameter through $P$ in a point $M$ on the directrix. The pencil $OQ$ is projective with the pencil $SM$, therefore with the range $M$ on the directrix, therefore with the pencil formed by joining $M$ to the point at infinity on the axis. Therefore the locus of $Q$ is a conic through $O$ and the point at infinity on the axis. A line through $O$ parallel to the directrix will meet a perpendicular tangent to the parabola at infinity. Thus a second point at infinity is obtained on the curve. Since these points are in perpendicular directions the locus is a rectangular hyperbola.

10. A variable diameter of a conic meets a fixed line in $P$ and through $P$ a line is drawn parallel to the conjugate diameter. Prove that the envelope of this line is a parabola.

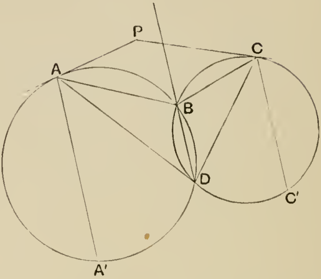
Let $V$ be the pole of the fixed line. Then the point at infinity $I$ on a parallel through $V$ to the conjugate diameter will be the pole of the diameter. Hence the ranges described by $P$ on the fixed line and by $I$ on the line at infinity are projective But the line whose envelope is required is $PI$ and the envelope is therefore a conic touching the line at infinity, that is a parabola.

11. Two points $P$ and $Q$ are conjugate with regard to a conic. $P$ lies on a fixed straight line and $PQ$ subtends a right angle at a fixed point. Prove that the locus of $Q$ is a conic passing through the fixed point.

Let $PM$ be the fixed line on which $P$ lies and $O$ the fixed point. The polar $QL$ of $P$ will pass through a fixed point $N$, the pole of $PM$. The range described by $P$ is projective with the pencil described by its polar $NQ$. Also the pencil $OQ$ is projective with the pencil $OP$ and therefore with the range $P$. Hence the pencils $OQ$ and $NQ$ are projective and the locus of $Q$ is a conic through $O$ and $N$.

12. $A$, $B$, $C$ are three fixed points. A circle is drawn through $A$, $B$ and another through $B$, $C$. If these two circles have a fixed radical axis, prove that their respective tangents at $A$, $C$ meet on a fixed conic.

Let $D$ be the second point of intersection of the circles. Through $A$ and $C$ draw $AA'$, $CC'$ parallel to $BD$. Then the pencils $AD$ and $CD$ for different positions of $D$ are projective. But since angle $A'AD = BDA = BAP$, the pencils $AD$ and $AP$ are projective.



Similarly the pencils $CD$ and $CP$ are projective. Therefore the pencils $AP$ and $CP$ are projective, and the locus of $P$ is a conic through $A$ and $C$.

13. Given a straight line along which the base of a triangle lies and an inscribed conic of the triangle, find the locus of the vertex if the base always subtends a right angle at a fixed point.

Let $ABC$ and $A'B'C'$ be two positions of the triangle. Since the base always subtends a right angle at a fixed point the ends of the base are a pair of conjugate points of an involution determined by $BC$ and $B'C'$. Hence the tangents from the ends of the base always intersect on the straight line $AA'$, which is therefore the locus of the vertex. (Arts. 75 and 95 (*c*).)

14. Through a fixed point $O$ a straight line is drawn to meet a straight line $l$ in $P$ and to intersect the polar of $P$ with respect to a fixed conic in $Q$. Show that as $P$ describes the straight line $l$, $Q$ describes a conic passing through two fixed points independent of the line $l$ chosen.

Let $O'$ be the pole of $l$, which will be situated on $p$ the polar of $P$. Then for different positions of $P$, the pencil described by $O'Q$ will be projective with the range described by $P$ and consequently with the pencil $OP$. Therefore the locus of $Q$, the point of intersection of corresponding rays of these pencils, is a conic through $O$ and $O'$. It also passes through the points of contact of the tangents from $O$ to the conic.

15. $A$, $B$, $C$ are three fixed points, $s$ a fixed line and $S$ a fixed conic passing through $A$, $B$; $Q$ is any point on $s$; $AQ$ meets the conic in $R$; show that the locus of the intersection of $CQ$, $BR$ is a conic passing through $B$, $C$, the intersections of $s$ and $S$ and through $D$ the other point of intersection of $AC$ and $S$.

Pencil $CQ$ is projective with the pencil $AR$ and therefore with the range $R$ on the conic. Therefore the pencils $BR$ and $CQ$ are projective and the locus of $P$ is a conic through $B$ and $C$. It obviously passes through the intersections of $s$ with $S$ and through $D$.

16. Two conics $AEFC$, $BEFD$ are such that the points $C$, $D$, $F$ are collinear; $L$ is a point on $EF$; $AL$ meets the first conic again in $Q$ and $BL$ meets the second conic again in $R$; show that the locus of the point of intersection of $CQ$, $DR$ is a straight line through $E$.

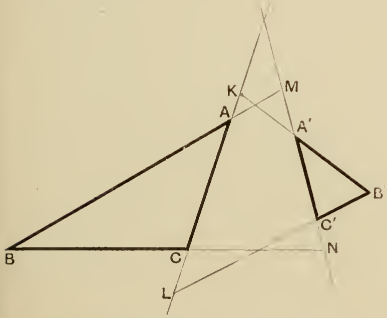
The pencils $AL$, $BL$ are projective. Therefore the ranges $Q$ and $R$ are projective, and the pencils $CQ$, $DR$ are projective. Since $CDF$ is a self-corresponding ray the ranges are in perspective and the locus of $U$ is a straight line which passes through $E$.

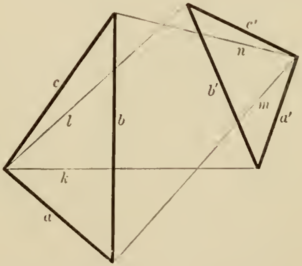
# CHAPTER XIX

## SIX POINTS ON A CONIC; SIX TANGENTS TO A CONIC

**119.** In Chapter XIII it was shown how a conic could be described through five given points or to touch five given straight lines. In certain special cases a conic may be described to pass through more than five points or to touch more than five lines.

(*a*) *If a conic can be described through the six vertices of two triangles, another conic can be described to touch the six sides, and correlatively, if a conic can be described to touch the six sides, another conic can be described through the six vertices of the triangles.*





Let $ABC$ and $A'B'C'$ be the two triangles and let $AC$ meet $B'A'$ and $B'C'$ in $K$ and $L$, and $A'C'$ meet $BA$ and $BC$ in $M$ and $N$.

If a conic circumscribes $A$, $B$, $C$, $A'$, $B'$, $C'$,

$$(B'.ACA'C') = (B.ACA'C').$$

Therefore taking intercepts on $AC$ and $A'C'$

$$(ACKL) = (MNA'C').$$

Hence $AM$, $CN$, $KA'$, $LC'$, $AC$ and $A'C'$ envelope a conic, which touches the sides of the two triangles.

Let $abc$ and $a'b'c'$ be the two triangles and let the connectors of $ac$ to $b'a'$ and $b'c'$ be $k$ and $l$, and those of $a'c'$ to $ba$ and $bc$ be $m$ and $n$.

If a conic touches $a$, $b$, $c$, $a'$, $b'$, $c'$,

$$(b'.aca'c') = (b.aca'c').$$

Therefore by joining the ranges to $ac$ and $a'c'$

$$(ackl) = (mna'c').$$

Hence $am$, $cn$, $ka'$, $lc'$, $ac$ and $a'c'$ all lie on a conic, which passes through the vertices of the two triangles.

(*b*) *A conic can be described to touch the six sides of two triangles or to pass through their six vertices, when the triangles are* (1) *self-conjugate with regard to a conic,* (2) *constructed of two pairs of tangents to a conic and their respective chords of contact,* (3) *circumscribed to a triangle and in perspective with it or* (4) *inscribed in a triangle and in perspective with it.*

Let $ABC$ and $A'B'C'$ be the triangles. Then in the figure (p. 259)

In case (1), $A$, $C$, $K$, $L$ are the poles of the lines joining $B$ to $C$, $A$, $C'$, $A'$ respectively.

In case (2), if $B$ and $B'$ be the poles of $AC$ and $A'C'$, $A$, $C$, $K$, $L$ are the poles of the lines joining $B$ to $A$, $C$, $A'$, $C'$ respectively.

Therefore in both cases

$$(B' . ACA'C') = (B . ACA'C')$$

and taking intercepts on $AC$ and $A'C'$

$$(ACKL) = (MNA'C').$$

Hence a conic can be described to touch $AM$, $CN$, $KA'$, $LC'$, $AC$ and $A'C'$, *i.e.* to touch the six sides of the triangles.

Case (3) follows from Menelaus' theorem and the converse of the correlative of Carnot's theorem.

Case (4) follows by (*a*) from the theorem on the right-hand side.

Let $abc$ and $a'b'c'$ be the triangles. Then in the figure (p. 259)

In case (1), $a$, $c$, $k$, $l$ are the polars of the intersection of $b$ with $c$, $a$, $c'$, $a'$ respectively.

In case (2), if $b$ and $b'$ be the polars of $ac$ and $a'c'$, $a$, $c$, $k$, $l$ are the polars of the intersections of $b$ with $a$, $c$, $a'$, $c'$ respectively.

Therefore in both cases

$$(b' . aca'c') = (b . aca'c')$$

and by joining the ranges to $ac$ and $a'c'$

$$(ackl) = (mna'c').$$

Hence a conic can be described through $am$, $cn$, $ka'$, $lc'$, $ac$ and $a'c'$, *i.e.* to pass through the six vertices of the triangles.

Case (3) follows by (*a*) from the theorem on the left-hand side.

Case (4) follows from Ceva's theorem and the converse of Carnot's theorem.

**120.** At the end of Art. 101 a short statement was given in regard to the properties of the conic involving the "imaginary." As there stated it is often assumed that a pair of real points can be replaced by, or projected into, a pair of imaginary points, these points usually being the circular points at infinity. No justification of this has been attempted and the consideration of this part of the subject is reserved for the present. The theorems however of Art. 119 afford instances of how this method may be applied. As such the following are given.

Particular case of $(a)$:

If two of the vertices of one triangle be assumed to be the circular points at infinity, then the following theorem is obtained.

*A parabola can be inscribed in any triangle, inscribed in a circle, so that its focus may be at any given point on the circumference of the circle.*

If two of the vertices of one triangle be assumed to be the circular points at infinity, then the following theorem is obtained.

*The circle, which circumscribes a triangle formed by three tangents to a parabola, passes through its focus.*

Particular cases of $(b)$ (1):

If two of the vertices of one of the self-conjugate triangles are assumed to be the circular points at infinity, the conic will be a rectangular hyperbola.

Hence *the conic inscribed in a triangle self-conjugate with regard to a rectangular hyperbola and having a focus at the centre of the rectangular hyperbola is a parabola.*

If two of the vertices of one of the self-conjugate triangles are assumed to be the circular points at infinity, the conic will be a rectangular hyperbola.

Hence *the circle circumscribing a triangle self-conjugate with regard to a rectangular hyperbola passes through the centre of the rectangular hyperbola.*

Particular cases of $(b)$ (2):

(i) If the two vertices of one triangle, in which the tangents meet the conic, be assumed to be the circular points at infinity, then the following theorem is obtained.

*The conic described to touch two tangents to a circle, and their chord of contact, and to have a focus at the centre, is a parabola.*

(ii) If one of the chords of contact be the line at infinity, then the following theorem is obtained.

*The conic which can be described to touch two tangents to a hyperbola, their chord of contact and the two asymptotes is a parabola.*

(i) If the two vertices of one triangle, in which the tangents meet the conic, be assumed to be the circular points at infinity, then the following theorem is obtained.

*The circle described through the points of contact of two tangents to a circle and through their point of intersection passes through the centre of the circle.*

(ii) If one of the chords of contact be the line at infinity, then the following theorem is obtained.

*If a hyperbola be described to pass through the points where any line meets a given hyperbola, to pass through the pole of this line, and to have its asymptotes parallel to those of the given hyperbola, then this hyperbola will pass through the centre of the given hyperbola.*

**121.** Certain of the theorems of Art. 119 may be expressed in a somewhat different form:

**If two conics are such that a triangle can be* $(a)$ *inscribed in one and circumscribed to the other,* $(b)$ *inscribed in one and self-conjugate with regard to the other, or* $(c)$ *circumscribed to one and self-conjugate with*

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* For similar theorems for a quadrangle see Art. 134.

*regard to the other, then an infinite number of such triangles can be constructed.*

The proofs in each case are similar. Let $ABC$ be the given triangle.

($a$) Let (1) be the conic to which $ABC$ is circumscribed. Draw $B'C'$ any tangent to this conic to meet the other conic (2) in $B'$, $C'$. Let the other tangents from $B'$ and $C'$ to (1) meet at $A'$. Then a conic can be described through $A$, $B$, $C$, $A'$, $B'$, $C'$. But (2) passes through $A$, $B$, $C$, $B'$, $C'$ and is therefore this conic, which also passes through $A'$.

($b$) Let (1) be the conic with regard to which the triangle $ABC$ is self-conjugate. Take $B'$ any point on the other conic (2), and let its polar with regard to (1) meet (2) in $C'$. Construct the polar of $C'$ with regard to (1). This passes through $B'$. Let these polars of $B'$ and $C'$ with regard to (1) intersect at $A'$. Then $ABC$ and $A'B'C'$ are two triangles self-conjugate with regard to (1). Hence a conic will pass through $A$, $B$, $C$, $A'$, $B'$, $C'$. But (2) which passes through $A$, $B$, $C$, $B'$, $C'$ must be this conic.

($c$) Let (1) be the conic with regard to which the triangle $abc$ is self-conjugate. Take $b'$ any tangent to the other conic (2) and let a tangent from its pole $B'$, with regard to (1), to (2) be $c'$. Let $a'$ be the polar of $b'c'$ with regard to (1). This passes through $B'$. Then $a'b'c'$ is self-conjugate with regard to (1). A conic can then be described to touch the sides of the triangles $abc$ and $a'b'c'$. As (2) touches five of these lines it must be this conic, and must therefore touch the sixth side $a'$.

It should be noticed that two triangles are in perspective

(1) when one triangle is the polar triangle of the other with regard to a conic (Art. 103 ($a$));

(2) when one triangle is self-conjugate with regard to a conic and the other is inscribed in the conic and circumscribed to the former triangle (Art. 107 ($b$)), or correlatively

when one triangle is self-conjugate with regard to a conic and the other is circumscribed to the conic and inscribed in the former triangle (Art. 107 ($d$));

(3) when one of the triangles is inscribed in and the other is circumscribed to the same triangle and both are in perspective with it (Example 15, Ch. VI).

| | |
|---|---|
| **122.** A conic may be described to touch | A conic may be described to pass through |
| (1) *The six lines joining the vertices of a triangle to the points where the opposite sides are met by a conic.* | (1) *The six points where the tangents from the vertices of a triangle to a conic meet the opposite sides of the triangle.* |
| The converse of the correlative to Carnot's Theorem. Art. 99. | The converse of Carnot's Theorem. Art. 99. |

(2) *The six lines joining the non-corresponding vertices of two triangles in perspective.*

The converse of Brianchon's Theorem. Art. 100.

(3) *The eight tangents to two conics at their four points of intersection.* (Art. 126 (*d*).)

(4) *The eight sides of two quadrilaterals which have the same diagonal triangle.* (Art. 126 (*a*).)

(5) *Any three pairs of lines through the diagonal points of a quadrangle, which are harmonic conjugates of the pairs of sides meeting at the diagonal points.*

(6) *Three pairs of lines* $m_1$, $n_1$; $m_2$, $n_2$; $m_3$, $n_3$, *where* $m_1$, $n_1$ *are the common harmonic conjugates of* $a_1$, $a_1'$ *and* $a_2$, $a_2'$—$a_1$, $a_2$ *and* $a_1'$, $a_2'$ *being the tangents from a vertex of a common self-conjugate triangle of two conics to the conics—and* $m_2$, $n_2$ *and* $m_3$, $n_3$ *are the similar lines for the other vertices of the common self-conjugate triangle.*

(7) *The six harmonic conjugates of the six lines joining any point to the vertices of a quadrilateral, with respect to the pairs of sides of the quadrilateral meeting at each vertex.* (Art. 117 (2).)

(8) *The six polars of a given point with respect to the six conics, which can be described through five out of six given points.* (Art. 135 (*b*).)

(2) *The six points of intersection of the non-corresponding sides of two triangles in perspective.*

The converse of Pascal's Theorem. Art. 100.

(3) *The eight points of contact of the common tangents to two conics.* (Art. 126 (*d*).)

(4) *The eight vertices of two quadrangles which have the same diagonal points triangle.* (Art. 126 (*a*).)

(5) * *Any three pairs of points on the diagonals of a quadrilateral, which are harmonic conjugates of the ends of the diagonals.*

(6) *Three pairs of points* $M_1$, $N_1$; $M_2$, $N_2$; $M_3$, $N_3$, *where* $M_1$, $N_1$ *are the common harmonic conjugates of* $A_1$, $A_1'$ *and* $A_2$, $A_2'$—$A_1$, $A_2$ *and* $A_1'$, $A_2'$ *being the points in which two conics are met by a side of their common self-conjugate triangle—and* $M_2$, $N_2$ *and* $M_3$, $N_3$ *are similar points for the other sides of the common self-conjugate triangle.*

(7) *The six harmonic conjugates of the six points in which the sides of a quadrangle are met by any straight line, with respect to the pairs of vertices of the quadrangle situated on each side.* (Art. 117 (2).)

(8) *The six poles of a given line with respect to the six conics, which can be described to touch five out of six given lines.* (Art. 135 (*b*).)

The proofs of (5) and (6) are as follows:

(5) Take the diagonal points triangle $ABC$ as triangle of reference. Let the ratios of $l$, $m$, $s$ in the figure be $a$, $b$, $c$. Then since these lines are concurrent $abc = -1$.

The ratios of $n$, $r$, $k$ in the figure are

(5) Take the diagonal triangle $ABC$ as triangle of reference. Let the ratios of $K$, $S$, $N$ in the figure be $a$, $b$, $c$. Then since these points are collinear $abc = +1$.

The ratios of $M$, $R$, $L$ in the figure therefore $-a$, $-b$, $-c$. If $x_1$, $x_2$ are harmonic conjugates of $a$, $-a$, then

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* This is practically a restatement of Art. 103 (*c*).

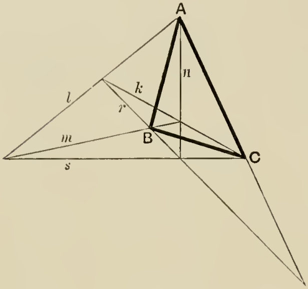
$$x_1x_2 = a^2.$$

Similarly $$y_1y_2 = b^2\,;$$

$$z_1z_2 = c^2.$$

$$\therefore\ x_1x_2y_1y_2z_1z_2 = a^2b^2c^2 = 1.$$

$\therefore$ by the converse of the correlative to Carnot's Theorem the six lines touch a conic.



(6) If the ratios of the points determined by $a_1$, $a_1'$ on the sides of the self-conjugate triangle are $a_1$, $a_1'$, then those of $a_2$, $a_2'$ will be $-a_1$ and $-a_1'$.

The common harmonic conjugates $(x_1, x_2)$ of these are given by

$$x_1 = -x_2 \text{ and } x_1x_2 = -a_1a_1'.$$

The condition that $x_1x_2y_1y_2z_1z_2$ should touch a conic is $x_1x_2y_1y_2z_1z_2 = 1$,

or $$a_1a_1'\,b_1b_1'\,c_1c_1' = -1 \quad \text{......(i)}.$$

But $$a_1{}^2b_1{}^2c_1{}^2 = -1,$$

and $$a_1'^2b_1'^2c_1'^2 = -1.$$

$$\therefore\ (a_1a_1'\,b_1b_1'\,c_1c_1')^2 = 1,$$

$$\therefore\ a_1a_1'\,b_1b_1'\,c_1c_1' = 1 \text{ or } -1.$$

Since the former does not generally hold, the latter is true and (i) is satisfied.

are therefore $-a$, $-b$, $-c$. If $x_1$, $x_2$ are harmonic conjugates of $a$, $-a$, then

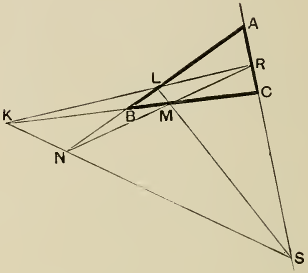
$$x_1x_2 = a^2.$$

Similarly $$y_1y_2 = b^2\,;$$

$$z_1z_2 = c^2.$$

$$\therefore\ x_1x_2y_1y_2z_1z_2 = a^2b^2c^2 = 1.$$

$\therefore$ by the converse of Carnot's Theorem the six points are on a conic.



(6) If the ratios of $A_1$, $A_1'$ referred to the self-conjugate triangle be $a_1$, $a_1'$, then those of $A_2$, $A_2'$ will be $-a_1$, $-a_1'$.

The common harmonic conjugates $(x_1, x_2)$ of these pairs of points are given by

$$x_1 = -x_2 \text{ and } x_1x_2 = -a_1a_1'.$$

The condition that $x_1x_2y_1y_2z_1z_2$ should lie on a conic is $x_1x_2y_1y_2z_1z_2 = 1$,

or $$a_1a_1'\,b_1b_1'\,c_1c_1' = -1 \quad \text{......(i)}.$$

But $$a_1{}^2b_1{}^2c_1{}^2 = -1,$$

and $$a_1'^2b_1'^2c_1'^2 = -1.$$

$$\therefore\ (a_1a_1'\,b_1b_1'\,c_1c_1')^2 = 1,$$

$$\therefore\ a_1a_1'\,b_1b_1'\,c_1c_1' = 1 \text{ or } -1.$$

Since the former does not generally hold, the latter is true and (i) is satisfied.

## EXAMPLES.

(1) $A$, $B$, $C$ and $D$ are any four points on a conic; $AB$ and $CD$ intersect in $O$, the tangents at $A$ and $B$ intersect at $P$ and the tangents at $C$ and $D$ at $Q$. Show that a conic can be described through the points $A$, $B$, $C$, $D$, $P$, $Q$ and that the tangents to this conic at $P$ and $Q$ are the straight lines $OP$, $OQ$.

By Art. 119 *b* (2), $A$, $B$, $C$, $D$, $P$, $Q$ are on a conic.

Since $PQ$ is the polar of $O$ with respect to this conic—which passes through $ABCD$—$OP$ and $OQ$ are the tangents to it from $O$.

(2) On the sides $BC$, $CA$ of a triangle $ABC$ points $A'$, $A''$ and $B'$, $B''$ are taken, and a point $C'$ is taken on the side $AB$; $A'B'$ and $C'B''$ meet in $H$, $A''B''$ and $BH$ meet in $K$, and $B'K$ meets $AB$ in $C''$. Prove that $AA'$, $AA''$, $BB'$, $BB''$, $CC'$, $CC''$ are tangents to a conic.

Consider the hexagon $A'A''B''C'C''B'$. The pairs of opposite sides intersect in the collinear points $B$, $K$, $H$. Therefore the vertices are on a conic. Therefore the conic through $A'A''B'B''C'$ passes through $C''$, and a conic can be described to touch the lines joining these points to the opposite vertices.

(3) $AB$ is a fixed chord of a conic and $CD$ is a fixed tangent at $C$ meeting $AB$ in $D$; $CP$ and $CQ$ are any two lines meeting the conic in $P$ and $Q$ and cut by any chord $l$ through $D$ in $P'$ and $Q'$. Show that a conic can be drawn through $ABPQP'Q'$.

Let $l$ meet the conic in $L$ and $M$ and $PQ$ in $R$: then $P'$, $Q'$ are conjugate points of the involution determined on $l$ by conics through $ABPQ$. For they are conjugate points of the involution determined by $CCPQ$, and for both involutions $D$, $R$ and $L$, $M$ are pairs of conjugate points.

Hence a conic through $A$, $B$, $P$, $Q$, $P'$ passes through $Q'$.

(4) $OPP'$, $OQQ'$, $OXX'$ are chords of a conic; prove that $PQ$, $PQ'$, $P'Q$, $P'Q'$ and the tangents at $X$, $X'$ all touch a conic.

Let $PQ' . QP'$ be $F$ and $PQ . P'Q'$ be $E$. Describe a conic to touch $PQ$, $PQ'$, $P'Q$, $P'Q'$ and the tangent at $X$. The two conics have $OEF$ for common self-conjugate triangle and are therefore in harmonic perspective with $O$ as centre and $EF$ as axis. The tangents at $X$ and $X'$ correspond, and, since the second conic touches the tangent at $X$, it also touches the tangent at $X'$.

(5) From any point on the circumscribed circle of a triangle $ABC$ perpendiculars are drawn to the sides $BC$, $CA$, $AB$ to meet the circle again in $D$, $E$, $F$ respectively. Prove that a conic can be drawn to touch the six straight lines $AB$, $BC$, $CA$, $DE$, $EF$, $FD$.

(6) A conic may be described through the six vertices of two triangles which are in harmonic perspective.

Let the triangles be $ABC$, $A'B'C'$ and $S$ the centre of perspective. Take $M$ and $N$ harmonic conjugates of $S$ with respect to $AA'$, $BB'$. $MN$ is then the axis of perspective. Describe a conic through $AA'$, $BB'$ and $C$. This conic is in self-perspective (harmonic) with $S$ and $s$ as centre and axis. Therefore the conic passes through $C'$.

Conversely, if two triangles are in perspective and a conic can be described through their vertices, the triangles are in harmonic perspective.

# CHAPTER XX

## THEOREMS CONCERNING TWO CONICS:—COMMON CONJUGATE POINTS, CONICS IN SELF-PERSPECTIVE, CONICS IN PERSPECTIVE WITH EACH OTHER, CONICS WITH DOUBLE CONTACT, HARMONIC LOCUS OF TWO CONICS

## TWO CONICS.

### 123. RECAPITULATION.

*Conjugate Points.*

A Conic determines on every straight line in its plane an involution, any pair of conjugate points of which are conjugate points with regard to the conic, while its double points, if real, are the points of intersection of the line and conic. (Art. 95 ($k$).)

The conjugate $A'$ on a given line $l$ of any point $A$ on the line $l$ with respect to a given conic may be constructed—

(1) as the point of intersection of the polar of $A$ with the line $l$;

(2) as the harmonic conjugate of $A$ with regard to the points $E$ and $F$ (if real), in which $l$ meets the conic;

(3) by taking any chord $PQ$ of the conic which passes through the pole of $l$, joining $A$ to $P$ to meet the conic at $R$, and joining $R$ to $Q$ to meet $l$ at $A'$.

(Art. 95 ($j$).)

*Conjugate Lines.*

A Conic determines at every point in its plane an involution pencil, any pair of conjugate rays of which are conjugate lines with regard to the conic, while its double rays, if real, are the tangents from the point to the conic.

(Art. 95 ($k$).)

The conjugate $a'$ through a given point $L$ of any ray $a$ through $L$ with respect to a given conic may be constructed—

(1) as the connector of the pole of $a$ with the point $L$;

(2) as the harmonic conjugate of $a$ with regard to the tangents $e$ and $f$ (if real) from $L$ to the conic;

(3) by taking any pair of tangents $p$, $q$ to the conic which intersect on the polar of $L$, constructing $r$ the tangent from $ap$ to the conic and constructing $a'$ the line joining $rq$ to $L$.

(Corl. of Art. 95 ($j$).)

*Common Conjugates of Conics through four points and of conics touching four straight lines.*

If a pair of points are conjugate points with regard to any two conics (including a pair of lines as a conic), they are conjugate points with regard to every conic through their four points of intersection. (Art. 117 (1).)

If a pair of lines are conjugate lines with regard to any two conics (including a pair of points as a conic), they are conjugate lines with regard to every conic which touches the four common tangents to these conics. (Art. 117 (1).)

### *Construction of Common Conjugates.*

(1) To construct the common conjugate of a given point $A$ with respect to a system of conics through four given points.

Construct the polars of $A$ with respect to two of the conics. Their point of intersection $A'$ is the common conjugate of $A$.

(2) To construct on a given line $l$ a pair of points, which are common conjugates with respect to a system of conics through four fixed points.

Construct on $l$ the involutions determined on this line by two of the conics. The pair of common conjugates of these involutions are the required points.

Or

Construct the two conics of the system which touch the line $l$. Their points of contact are the pair of common conjugates on $l$.

If the points $A$, $B$, $C$, $D$, through which the conics pass, are determined as the intersections of a pair of lines $AB$ and $CD$ with a conic $S$, the following construction is useful:

Draw the tangent $p$ at any point $P$ on the conic. Join $P$ to $AB.CD$ by $a$. Take $a'$ the harmonic conjugate of $a$ with respect to $AB$ and $CD$. $a'$ meets $p$ in $P'$ a common conjugate of $P$.

This follows from the fact that every point on the tangent at $P$ is a conjugate of $P$ with regard to the conic.

The connectors of a pair of points, constructed in this way, with the diagonal points of the quadrangle are harmonic conjugates of the pairs of sides which meet at these points.

(1) To construct the common conjugate of a given line $a$ with respect to a system of conics touching four given lines.

Construct the poles of $a$ with respect to two of the conics. Their connector $a'$ is the common conjugate of $a$.

(2) To construct through a given point $L$ a pair of lines, which are common conjugates with respect to a system of conics touching four fixed lines.

Construct the involution pencils determined at $L$ by two of the conics. The pair of common conjugates of these involution pencils are the required lines.

Or

Construct the two conics of the system which pass through $L$. The tangents at $L$ to these conics are the pair of common conjugates through $L$.

If the lines $a$, $b$, $c$, $d$, which are touched by the conics, are determined as the tangents for a pair of points $ab$ and $cd$ to a conic $S$, the following construction is useful:

Construct the point of contact $P$ of any tangent $p$ to the conic. Construct $A$ the point of intersection of $p$ and $ab.cd$. Take $A'$ the harmonic conjugate of $A$ with respect to $ab$ and $cd$. $p'$ the connector of $A'$ to $P$ is a common conjugate of $p$.

This follows from the fact that every line through the point of contact of a tangent $p$ is a conjugate of $p$ with regard to the conic.

The points of intersection of pairs of lines, constructed in this way, with the diagonals of the quadrilateral are harmonic conjugates of the pairs of vertices on these diagonals.

*Eleven points locus and its correlative.*

Given a system of conics through four fixed points the locus of the common conjugates of points on a fixed line is a conic which passes through eleven determined points.

This important theorem is proved in Art. 117 (2).

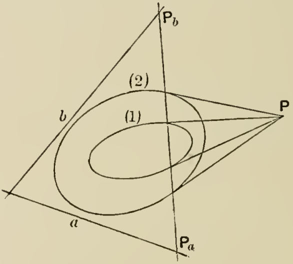
The polars of any two fixed points with respect to a system of conics through four fixed points form two projective pencils. (Art. 106 (*e*).)

Given a system of conics touching four fixed lines the envelope of the common conjugates of lines passing through a fixed point is a conic which touches eleven determined lines.

This important theorem is proved in Art. 117 (2).

The poles of any two fixed lines with respect to a system of conics touching four fixed lines form two projective ranges. (Art. 106 (*e*).)

**124.** *Every pair of real conics have a real common self-conjugate triangle except when the conics intersect in only two real points, in which case two vertices and the opposite sides of the triangle are imaginary.*



The loci of the common conjugates of points on lines $a$ and $b$ with respect to the conics (1) and (2) are two conics.

The common conjugate of the point $ab$ must be on both of these conics. These conics will therefore intersect in another real point $P$. Let $P_a$ and $P_b$ be the common conjugates of $P$ on the lines $a$ and $b$. Since $P_a$ and $P_b$ are conjugates of $P$ with respect to (1), $P_aP_b$ is the polar of $P$ with respect to (1).

Similarly it is the polar of $P$ with respect to (2). Therefore $P_aP_b$ is a common polar of $P$.

The chord $P_aP_b$ may meet the conics

(1) in real points $A_1$, $B_1$ and $A_2$, $B_2$ such that $A_1B_1$ and $A_2B_2$ do not overlap.

(2) in real points $A_1$, $B_1$ and $A_2$, $B_2$ such that $A_1B_1$ and $A_2B_2$ do overlap.

(3) in real points $A_1$, $B_1$ and in two imaginary points.

(4) in four imaginary points.

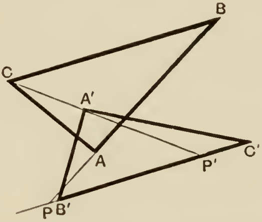
The vertices of the common self-conjugate triangle situated on $P_aP_b$ are the common harmonic conjugates of $A_1B_1$ and $A_2B_2$, *i.e.* the double elements of the involution which they determine.

In case (1) these points are real, (Art. 48)
(2) these points are imaginary conjugate points, „
(3) these points are real, „
(4) these points are real. „

In case (2) the conics intersect in two real and two imaginary points.

*Two conics which intersect in four distinct points can have only one common self-conjugate triangle.*

If possible let $ABC$ and $A'B'C'$ be two common self-conjugate triangles. Then $B'$ and $C'$ are conjugate points with respect to both conics. Let $AB$ meet $B'C'$ in $P$.



Then the polar of $P$ with regard to both conics is $CA'$. Let $CA'$ meet $B'C'$ in $P'$. Then $P$ and $P'$ are common conjugate points with respect to the two conics. Hence, since $B'$, $C'$ and $P$, $P'$ are common conjugate points the conics must meet $B'C'$ in the same pair of points. Similarly they meet the other five sides of the triangles in the same points. Hence they have twelve points common and are therefore the same conic.

### 125. Common involution chords of two conics.

*There are associated with every pair of conics two real lines on which the conics determine the same involution.*

If the conics have four real points of intersection, the lines in question are any two of their common chords.

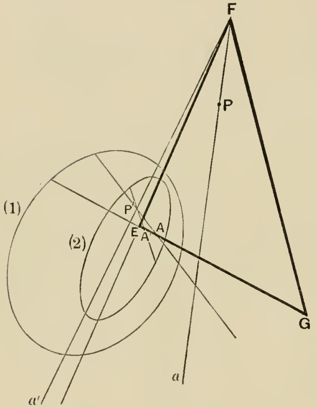
If the conics have no real points of intersection, consider the locus of the common conjugates of points on a line $a$, which passes through one of the vertices $F$ of

*There are associated with every pair of conics two real points at which the conics determine the same involution.*

If the conics have four real common tangents, the points are any two of the points of intersection of these common tangents.

If the conics have no real common tangents, consider the envelope of the common conjugates of lines through a point $A$ on one of the sides $f$ of the common the common self-conjugate triangle of the conics (1) and (2).

Let the poles of the line $a$ with regard to (1) and (2) be $A$ and $A'$. These points are on $EG$, where $FEG$ is the common self-conjugate triangle. The polars of $P$ any point on $a$ with respect to (1) and (2) will meet in some point $P'$.

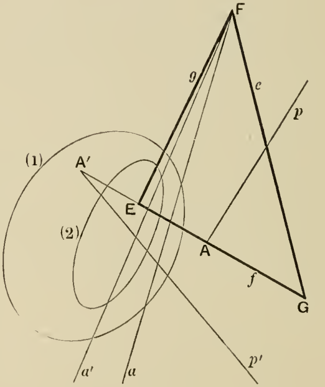


Since the points $A$, $A'$ and also the points $E$, $G$, $F$ are on the conic, which is the locus of $P'$, this conic will break up into a pair of lines, one of which is $EA'AG$ and the other $FP'$.

The polars of $P'$ with respect to (1) and (2) will pass through $P$. Therefore, if $FP'$ be $a'$, the locus of $P$ for different positions of $P'$ on $a'$ is the line $a$. Hence the lines $a$ and $a'$ form an involution

self-conjugate triangle of the conics (1) and (2).

Let the polars of the point $A$ with regard to (1) and (2) be $a$ and $a'$. These pass through $F$, where $FEG$ is the common self-conjugate triangle. The poles of $p$ any line through $A$ with respect to (1) and (2) will have as their connector some line $p'$.



Since the lines $a$, $a'$ and also the lines $e$, $g$, $f$ are tangents to the conic, which is the envelope of $p'$, this conic will break up into a pair of points, one of which is $ea'ag$ and the other $fp'$.

The poles of $p'$ with respect to (1) and (2) will be on $p$. Therefore, if $fp'$ be $A'$, the envelope of $p$ for different positions of $p'$ through $A'$ is the point $A$. Hence the points $A$ and $A'$ form an involution.

pencil. If $FG$ is taken as $a$, then $EF$ will be $a'$. Therefore $FG$ and $FE$ are conjugate elements of the involution. There will be a pair of double rays of this involution, which will be the common harmonic conjugates of $FE$, $FG$ and of $FP$, $FP'$. For these double rays $F$, $P$, $P'$ will be collinear. On these double rays, $p$ and $p'$, the involution determined by the two conics will be the same. If the intersections of the conics are real, these double rays $p$ and $p$ pass through the points common to (1) and (2), *i.e.* they are chords of intersection.

The condition that $p$ and $p'$ should be real is that the lines joining $P$ and $P'$ to the corresponding vertex of the triangle $EFG$ should not overlap the sides. There always will be one vertex for which they do not overlap. For, taking $P$ a point inside the triangle, $P'$ must lie inside one of the angles $EFG$, $FGE$ or $FEG$. If it lies inside $EFG$, then the pair of lines $FP$, $FP'$ do not overlap $EF$, $FG$.

If the conics have two real points of intersection, two vertices $E$ and $G$ of the triangle $EFG$ are imaginary, and one chord of intersection is real. This is one of the

If $fg$ is taken as $A$, then $ef$ will be $A'$. Therefore $fg$ and $fe$ are conjugate points of the involution. There will be a pair of double points of this involution, which will be the common harmonic conjugates of $fe$, $gf$ and $pf$, $p'f$. For these double points $f$, $p$, $p'$ will be concurrent. At these double points, $P$ and $P'$, the involution determined (by pairs of conjugate lines) by the two conics will be the same. Hence the double points $P$ and $P'$ are the points of intersection of the common tangents, if real, to the two conics.

The condition that $P$ and $P'$ should be real is that the points where the lines $p$ and $p'$ meet the sides of the triangle $efg$ should not overlap the vertices. There will always be one side for which they do not overlap. For, if $x_1$, $y_1$, $z_1$ be the ratios of the points where one line meets the sides of $EFG$ and $x_2$, $y_2$, $z_2$ be the corresponding ratios for the other, then

$$x_1y_1z_1 = 1\ ;\quad x_2y_2z_2 = 1.$$

Hence two or none of the ratios $x_1$, $y_1$, $z_1$ and $x_2$, $y_2$, $z_2$ must be negative; therefore one pair have always the same sign.

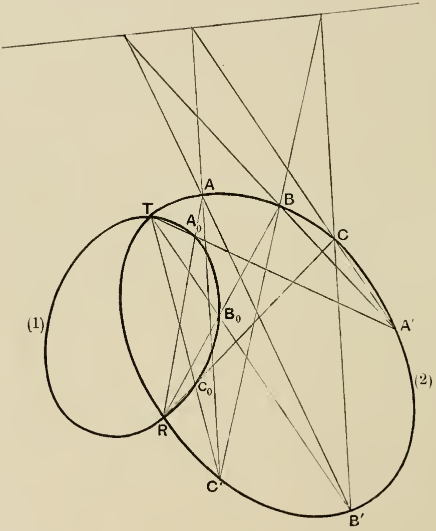
If the conics have two real common tangents, two sides of the triangle $efg$ are imaginary, and the points of intersection of these give one of the required points, which double rays of the involution $a$, $a'$ determined at $F$. The other double ray is the second chord.

Hence every pair of conics have one pair of real lines on which the conics determine the same involution.

is a double point of the involution $A$, $A'$. The other double point is the second required point.

Hence every pair of conics have a pair of real points at which the conics determine the same involution pencil.

*Given two real points of intersection of two conics, determining one chord of intersection, to construct a second chord on which the conics determine the same involution.*



Let $R$ and $T$ be the given points of intersection of the conics (1) and (2). Take any three points $A_0$, $B_0$, $C_0$ on (1). Let the connectors of these points to $R$ and $T$ meet the conic (2) in $A$, $B$, $C$ and $A'$, $B'$, $C'$ respectively. Then $A$, $B$, $C$ and $A'$, $B'$, $C'$ determine two projective ranges on (2) and the Pascal line of $AB'CA'BC'$ meets the conic (2) in the self-corresponding points of the two ranges. These points if joined to $R$ and $T$ give two pairs of corresponding rays of the pencils $(R \,.\, A_0B_0C_0)$ and $(T \,.\, A_0B_0C_0)$. Therefore they are on the conic (1). If as in the figure the points of intersection of this Pascal line with the conics are imaginary, the Pascal line is the second chord, on which the conics determine the same involution.

This theorem may be stated as follows :

*If $R$ and $T$ are two points of intersection of two conics and any two points $A_0$ and $B_0$ on one conic be projected upon the other conic into points $A$, $B$ and $A'$, $B'$ from $R$ and $T$ respectively, then the locus of $AB' . BA'$ is a common chord of the conics.*

A second construction for this chord is given in Art. 127.

### Construction of the common self-conjugate triangle of two conics.

In certain cases the common self-conjugate triangle of two conics may be easily constructed.

(1) If the conics intersect in four real points, the common self-conjugate triangle is the diagonal points triangle of the quadrangle formed by these four points. (Art. 95 ($p$).)

(2) If the conics have four real common tangents, the common self-conjugate triangle is the diagonal triangle of the quadrilateral formed by the four common tangents. (Art. 95 ($p$).)

(3) If the conics intersect in two real points $A$ and $B$, construct the poles $K$ and $K'$ of $AB$ with respect to the conics. Let $KK'$ meet $AB$ in $R$ and the conics in $P$, $Q$ and $P'$, $Q'$. Take $T$ the harmonic conjugate of $R$ with respect to $A$ and $B$. Then $T$ is the pole of $KK'$ with respect to both conics, and is therefore a vertex of the common self-conjugate triangle. The other vertices are the two imaginary common harmonic conjugates of $PQ$ and $P'Q'$.

(4) If the conics have two real common tangents, the correlative method to that employed in (3) may be used.

(5) If the conics have no real points of intersection and no real common tangents, the general method must be employed.

Certain of the points of intersection $A$, $B$, $C$, $D$ may coincide :

(1) If $D$ and $C$ coincide and $A$ and $B$ are real, the self-conjugate triangle constructed as the diagonal points triangle of $ABCD$ becomes in the limit the tangent $SD$, where $S$ is the point in which $AB$ meets the tangent at $D$. The point $S$ has a common polar with respect to the two conics, viz., the line joining $D$ to the harmonic conjugate of $S$ with respect to $A$ and $B$.

(2) If $D$ and $C$ coincide and $A$ and $B$ are imaginary, the line $AB$ may be constructed, and $S$ is thus determined. The conjugate of $S$ in the involution on $AB$, when joined to $D$, gives the polar of $S$.

(3) If $D$ and $C$ coincide and also $A$ and $B$, the conics have double contact. If the tangents at $A$ and $D$ meet in $S$, then $S$ and any pair of harmonic conjugates of $A$ and $D$ on $AD$ form a common self-conjugate triangle of the conics.

(4) If $B$, $C$, $D$ all coincide so that the conics have contact of the third order, the self-conjugate triangle in the limit coincides with the point $B$.

## EXAMPLES.

(1) Given three common points of two conics and a common pair, $A$ and $B$, of conjugate points, make a geometrical construction for the fourth common point.

Let $K$, $L$, $M$, $N$ be the points of intersection of the conic of which $N$ has to be found. If $A$, $B$ be conjugate points with respect to the two conics, they are conjugate points with respect to every conic through $K$, $L$, $M$, $N$ and therefore with respect to each pair of lines through these points. Hence, if the line $AB$ meets $KL$ and $MN$ in $A'$ and $B'$, $(ABA'B') = -1$. Therefore $B'$ is found and the line $MN$ is obtained. Similarly the line $KN$ may be found and therefore $N$ is known.

(2) Three of the intersections of two conics are given and two other points on each: to find the remaining point of intersection of the conics.

Let $A$, $B$, $C$ be the three given points of intersection. Construct $R$ and $R'$ the poles of $BC$ and $S$ and $S'$ the poles of $AB$. Let $SS'$ meet $BC$ in $K$ and $RR'$ meet $AB$ in $L$. Then $KA$ and $LC$ intersect in $D$ the fourth point of intersection of the conics.

(3) Two of the intersections of two conics are given and three other points on each: to find the remaining points of intersection of the conics.

Let $A$, $B$ be the given points of intersection: $P$, $Q$, $R$ the three given points on conic (1) and $U$, $V$, $W$ those on (2). Join $P$, $Q$, $R$ to $A$ and $B$ and find by Pascal's Theorem the points $P'$, $Q'$, $R'$ and $P''$, $Q''$, $R''$ where these lines meet conic (2). Then the points $P'Q''.Q'P''$ and $P'R''.P''R'$ determine the second common chord of intersection. Determine the points where this line meets either conic. These are the two other points of intersection of the conics.

(4) If two conics touch at $D$ and a chord $DRR'$ be drawn through $D$ to meet the conics in $R$ and $R'$, show that the locus of the point of intersection of the tangents at $R$ and $R'$ is a straight line through the other points of intersection of the conics.

(5) If two conics intersect in $A$, $B$, $C$, $D$, and through $A$ and $B$ two lines $a$ and $b$ be drawn to intersect on the line joining the points of intersection of the tangents to the conics at $A$ and $B$, then a third conic can be described to pass through $A$, $B$, $C$, $D$ and touch $a$ and $b$ at $A$ and $B$.

Draw a conic through $A$, $B$, $C$, $D$ to touch $a$ at $A$. Then in the harmonic perspective in which $AB.CD$ is the centre and its polar the axis, this conic will correspond to itself and will touch $b$, the line corresponding to $a$, at $B$, the point corresponding to $A$.

### Conics in Self-Perspective.

**126.** *Each of two conics may be looked upon as in harmonic perspective with itself, the centre and axis of perspective being a vertex and the opposite side of their common self-conjugate triangle.*

In Art. 95 ($o$) it was proved that any conic may be looked upon as in perspective with itself, if any point and its polar are taken as the centre and axis of perspective and the anharmonic ratio of the perspective is $-1$.

In Art. 124 it was shown that every pair of conics have a common self-conjugate triangle except when they intersect in two real and two imaginary points, in which case only one vertex and side are real. Hence, if a vertex and the opposite side of this triangle be taken as centre and axis of perspective and the anharmonic ratio of the perspective be $-1$, each of the conics is in perspective with itself.

As a particular case, the two pairs of sides of a quadrilateral are each in self-perspective with a vertex and opposite side of the diagonal triangle for the centre and axis of perspective and, correlatively, the two pairs of opposite vertices of a quadrangle are in self-perspective with a vertex and opposite side of the diagonal points triangle for centre and axis.

*Deductions.*

(*a*) *If two quadrilaterals have the same diagonal triangle, a conic can be described to touch their eight sides.*

Describe a conic to touch the four sides of one of the quadrilaterals and one of the sides of the other. The two quadrilaterals and the conic may be looked upon as all in self-perspective with any vertex of the diagonal triangle and the opposite side as centre and axis of perspective. Considering one vertex and the opposite side as centre and axis of perspective, it is seen that because the conic touches one side of the second quadrilateral it must touch a second. Similarly by considering the other two possible perspectives it follows that this conic also touches the other two sides of the second quadrilateral.

The correlative theorem is as follows:

*If two complete quadrangles have the same diagonal points triangle, a conic may be described through their eight vertices.*

(*b*) *All conics with respect to which a given triangle is self-conjugate and which pass through a given point pass through three other fixed points.*

The three other points are obtained from the given point by the three systems of self-perspective derived from the self-conjugate triangle.

(*c*) *To construct a conic with a given self-conjugate triangle to pass through two fixed points.*

From each of the two fixed points construct three other points by the three systems of self-perspective. A conic through the two given points and these six points is the required conic.

(*d*) *A conic can be described to touch the four pairs of tangents which can be drawn to two conics at their four points of intersection.*

Describe a third conic to touch the four tangents to one conic (1) and one of the tangents to the other conic (2). This conic and the two given conics have a common self-conjugate triangle. In the three systems of self-perspective the tangents to the two given conics at their points of intersection are corresponding lines. Hence, as the third conic touches one of the four tangents to the conic (2), it must touch them all.

The correlative theorem is as follows:

*A conic can be described through the four pairs of points of contact of the four common tangents to two conics.*

This may be proved in a similar manner.

### EXAMPLES.

(1) If, through a given point $S$ any fixed chord be drawn, show that tangents to any given conic, whose points of contact are collinear with $S$, determine conjugate points of an involution on the fixed chord.

(2) Show that pairs of common tangents to conics through four given points determine an involution on any chord through one of the points of intersection of a pair of common chords of the system.

(3) Prove that if two variable chords $SAA'$, $SBB'$ be drawn to a conic through a fixed point $S$, then the lines $AB$, $A'B'$ determine corresponding points of an involution on any chord through $S$.

(4) If two conics are inscribed in the same quadrilateral, the eight points of contact lie on a third conic, and the polars of a common chord of the first two conics with respect to all three are collinear.

### Conics in Perspective with each other.

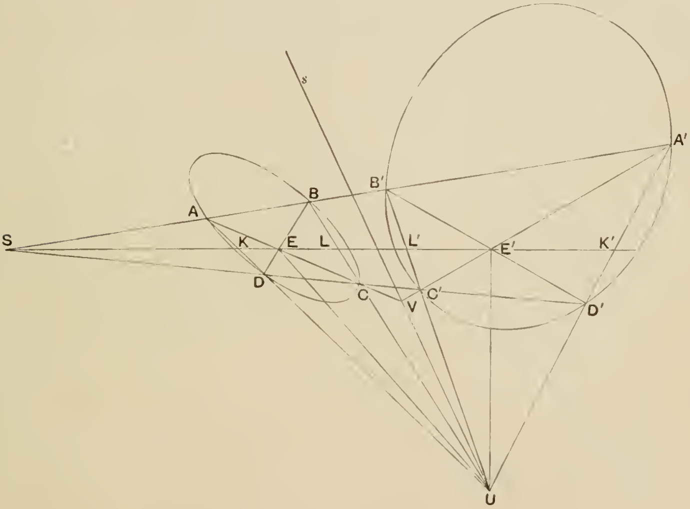
**127.** *Each of any two conics may be looked upon as the perspective of the other, either of the points at which the conics subtend the same involution being the centre of perspective, and either of the chords, on which they determine the same involution, being the axis of perspective.*

Through $S$ a common involution point draw any chord to meet the conics in $A$, $B$, $B'$ and $A'$, and let the polars of $S$ meet at $U$. Join $U$ to $A$, $B$, $B'$, $A'$ to meet the conics again in $D$, $C$, $C'$, $D'$. Let $AC . BD$ and $A'C' . B'D'$ be $E$ and $E'$. Then since $SE$ and $SE'$ are the polars of $U$ and $S$ is a common involution point $S$, $E$, and $E'$ are collinear. Let

$SEE'$ meet $UA$, $UB$, $UB'$ and $UA'$ in $K$, $L$, $L'$ and $K'$. Then since $(UKDA)$, $(ULCB)$, $(UL'C'B')$, $(UK'D'A')$ are all harmonic, $D$, $C$, $C'$, $D'$ are all collinear with $S$. Let $AC$ and $A'C'$ meet in $V$ and let $UV$ be $s$.

Consider the perspective with centre $S$, axis $s$ and $A$ and $A'$ for corresponding points. In this perspective $A$, $E$, $C$ and $D$ correspond to $A'$, $E'$, $C'$ and $D'$. Therefore $B$ corresponds to $B'$ and $EU$ to $E'U$. Hence the conics have four pairs of corresponding points.

Also $EU$ and $E'U$ are the polars of $S$. Hence if a pair of conjugate points with respect to the conic $ABCD$ be taken on $EU$, they will be



projected from $S$ into a pair of points on $UE'$ which are conjugate with respect to the conic $A'B'C'D'$. Therefore the involutions determined by the conics on these lines are projected into each other from $S$.

Hence the perspective of the conic $ABCD$ is the conic $A'B'C'D'$.

If the common tangents from the common involution point are real, the proof may be put into the following simpler form.

Let the common tangents $PP'$, $TT'$ to the two conics intersect at $S$. Through $S$ draw a chord meeting the conics in $Q$ and $Q'$. Then $PQT$ and $P'Q'T'$ are two triangles in perspective and therefore the corresponding sides will intersect on a line $s$. Take $S$ for centre of perspective and $s$ for axis. Then for the two conics

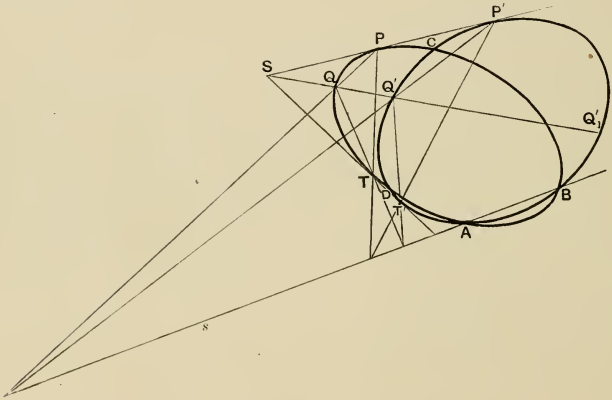
Two points at $P$ correspond to two points at $P'$.

Two points at $T$ correspond to two points at $T'$.

One point at $Q$ corresponds to one point at $Q'$.

Hence, since five points determine a conic, the conics are in perspective.

If $Q$ and $Q_1'$ in the figure are taken as corresponding points the conics are also in perspective, and the axis of perspective is the line joining the points of intersection of the corresponding sides of the triangles $PQT$ and $P'Q_1'T'$. Thus there are two perspectives with $S$ as centre.

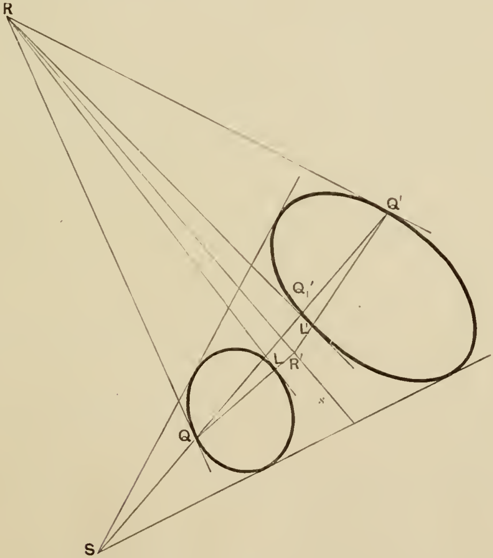


*Determination of the axes of perspective.*

If the conics have four common tangents and intersect in four real points as in the figure, let the point $Q$ correspond to the point $Q'$. As the point $Q'$ moves along one conic, the point $Q$ will move along the other. When $Q'$ is at $D$, $Q$ will be at the point where $DS$ meets the other conic. When $Q'$ reaches $A$, $Q$ will be at the same point. Hence $A$ is a self-corresponding point in the perspective. Similarly $B$ is a self-corresponding point. Therefore $s$ the axis of perspective passes through $A$ and $B$ and is therefore a common chord.

If $Q_1'$ is taken as corresponding to $Q$, the points $D$ and $C$ are self-corresponding points and $DC$ is the axis of perspective.

If the conics do not intersect in real points, let the point $Q$ correspond to the point $Q'$. Then the tangents at $Q$ and $Q'$, being corresponding lines, will intersect at a point $R$ on $s$. Draw the other tangents $RL$ and $RL'$ from $R$ to the two conics. These are corresponding lines. Therefore the chords of contact $QL$ and $Q'L'$ are also corresponding lines and intersect in $R'$ on $s$. $R$ and $R'$ are a pair of common conjugate points on $s$ with respect to the conics. Similarly any number of pairs of common conjugate points may be determined on $s$. Hence the two conics determine on $s$ the same involution, and $s$ is one of the common involution chords of the conics, and passes through two of their imaginary points of intersection.



Similarly, if $Q$, $Q_1'$ be taken as corresponding to each other, a second real chord passing through two other imaginary points of intersection of the conics is obtained as the axis of perspective.

Generally there are six points of intersection of the four common tangents and six common chords joining the four points of intersection of the two conics.

Corresponding to each of the six centres of perspective there are two axes of perspective, making in all twelve perspectives.

Each of the six centres of perspective has two axes of perspective and each of the six axes has two centres of perspective.

*The chords of contact of the common tangents to two conics pass four by four through the three vertices of their common self-conjugate triangle, that is, through the diagonal points of the quadrangle formed by their four points of intersection.*

In the figure, page 278, the chords of contact $PT$, $P'T'$ of the common tangents intersect on the axis of perspective corresponding to $S$, *i.e.* on the chord $AB$, when $AB$ is the axis of perspective. They must also intersect on $CD$, when $S$ and $CD$ are taken as the centre and axis of perspective. They, therefore, pass through $CD . AB$.

If the other pair of common tangents are drawn to intersect in $S'$, where $S'$ is the other centre of perspective which has $AB$ and $CD$ for axes of perspective, it follows in a similar way, that the chords of contact of this pair of common tangents pass through the point $CD . AB$.

### 128. Common inscribed quadrangle and circumscribed quadrilateral of two conics.

*If $A$, $B$, $C$, $D$ be the points of intersection of two conics* (1) *and* (2) *and $a$, $b$, $c$, $d$ their four common tangents, then*

(1) *the common self-conjugate triangle $EFG$ may be constructed as the diagonal points triangle of the quadrangle $ABCD$ or as the diagonal triangle of the quadrilateral $abcd$;*

(2) *the chords of contact of the common tangents $a$, $b$, $c$, $d$ with the two conics pass four by four through the three vertices $E$, $F$, $G$ of the common self-conjugate triangle; and correlatively*

(3) *the points of intersection of the tangents at $A$, $B$, $C$, $D$ lie four by four on the sides $e$, $f$, $g$ of the common self-conjugate triangle.*

$A$, $B$, $C$, $D$ determine a common self-conjugate triangle $EFG$ as their diagonal points triangle, and $a$, $b$, $c$, $d$ determine a common self-conjugate triangle $efg$ as their diagonal triangle. As there can be only one common self-conjugate triangle (Art. 124), $e$, $f$, $g$ are the sides of $EFG$.

The chords of contact of $a$, $b$, $c$, $d$ with conic (1) pass through the vertices of the diagonal triangle of $abcd$, that is through $E$, $F$, $G$. (Art. 80 ($a$).)

Similarly the chords of contact of $a$, $b$, $c$, $d$ with conic (2) pass through $E$, $F$, $G$.

The tangents at $A$, $B$, $C$, $D$ to conic (1) intersect in pairs on the sides of the diagonal points triangle of $ABCD$, *i.e.* on $e, f, g$. (Art. 80 ($a$).)

Similarly the tangents at $A$, $B$, $C$, $D$ to conic (2) intersect in pairs on $e, f, g$.

Since the conics are in self-perspective with any vertex of the self-conjugate triangle for centre and the opposite side for axis of perspective, all the pairs of corresponding points on the sides of the common self-conjugate triangle are harmonic conjugates with respect to the vertices on that side, and form an involution with these points as double points.

Let the vertices of $abcd$ on the sides $FG$, $GE$ and $EF$ of the triangle $EFG$ be $E_1$, $E_2$, $F_1$, $F_2$, $G_1$, $G_2$ respectively.

The tangents at the points where a line through $G_1$ meets the two conics intersect on $AD$ or $BC$. This follows from the fact that the conics are in perspective with each other, with any vertex $G_1$ of the circumscribed quadrilateral as centre of perspective and either of the common chords through the vertex of the common self-conjugate triangle opposite to the side on which $G_1$ is situated, as axis.

**A system of conics with a common self-conjugate triangle.**

A system of conics with a given common self-conjugate triangle $EFG$ is a doubly infinite system. For a conic can be described through any two points $A$ and $A'$ to have $EFG$ for a self-conjugate triangle. From $A$ three points $B$, $C$, $D$ can be deduced by harmonic perspective. Any conic through $ABCD$ has $EFG$ for a self-conjugate triangle. Similarly from $A'$ three other points $B'$, $C'$, $D'$ can be deduced by harmonic perspective. The two quadrangles $ABCD$ and $A'B'C'D'$ have a common self-conjugate triangle $EFG$ and their vertices—Art. 126 ($a$)—lie on a conic.

Similarly a conic can be described to touch any two lines $a$ and $a'$ and have $efg$ for self-conjugate triangle. From $a$ and $a'$ two quadrilaterals can be obtained by harmonic perspective and their eight sides $a$, $b$, $c$, $d$, and $a'$, $b'$, $c'$, $d'$ are all tangents to the conic.

Two conics can likewise be described to pass through any point $A$, to touch any line $a$ and to have $EFG$ for a self-conjugate triangle. From $A$ three other points $B$, $C$, $D$ can be deduced by harmonic perspective. Every conic through these points has $EFG$ for common self-conjugate triangle. Two conics can be described through $A$, $B$, $C$, $D$ to touch the line $a$ (Art. 113 ($D$)). These conics also touch the three lines $b$, $c$, $d$ obtained by harmonic perspective from $a$. Such a pair of conics have $a$, $b$, $c$, $d$ for common tangents and $A$, $B$, $C$, $D$ for their points of intersection.

*If a system of conics has a common self-conjugate triangle, any straight line through a vertex of this triangle is cut by the system of conics in pairs of conjugate points of an involution.*

If a line through a vertex $S$ of the common self-conjugate triangle meet the opposite side in $N$ and the conics in pairs of points $AA'$, $BB'$, $CC'$, ..., then

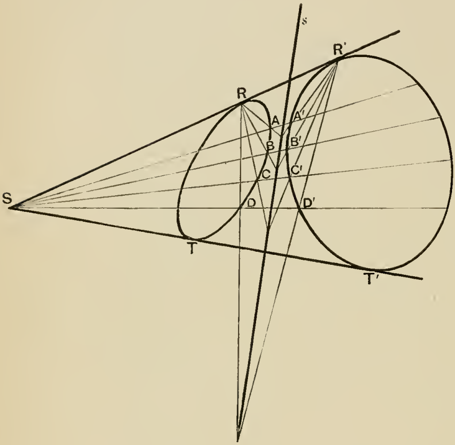
$$(SNAA')=(SNBB')=(SNCC')=-1.$$

Hence $AA'$, $BB'$, $CC'$, ... are conjugate points of an involution of which $S$ and $N$ are the double points.

The correlative theorem is as follows :

*If a system of conics has a common self-conjugate triangle and pairs of tangents are drawn from any point on one of the sides to the system of conics, these pairs of tangents are conjugate lines of an involution pencil.*

**129.** *If through a point of intersection of common tangents to two conics a variable chord be drawn to meet one conic in $A$ and the other in $A'$, then the ranges formed by $A$ and $A'$ on the two conics are projective.*

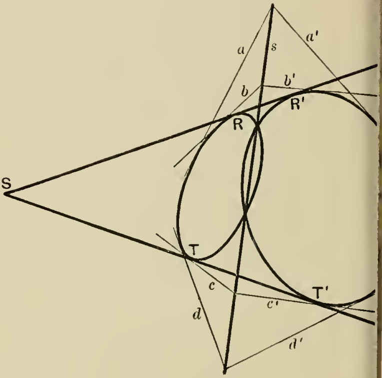


Let the common tangents $TT'$, $RR'$ intersect at $S$. Let $SAA'$, $SBB'$, $SCC'$, $SDD'$ be four positions of the variable chord. Join $A$, $B$, $C$, $D$ to $R$ and $A'$, $B'$, $C'$, $D'$ to $R'$.

Then the two conics are in perspective with each other, $S$ being the centre of perspective and the points $R$, $A$, $B$, $C$, $D$ corresponding to the points $R'$, $A'$, $B'$, $C'$, $D'$. (Art. 127.)

Therefore the lines $RA$, $RB$, $RC$, $RD$ correspond to $R'A'$,

*If from a variable point on a common chord of two conics a tangent $a$ be drawn to one and a tangent $a'$ to the other, then the systems of tangents to the two conics formed by $a$ and $a'$ are projective.*



Let $s$ be a common chord of the two conics and from points on $s$ draw tangents $a$, $b$, $c$, $d$ and $a'$, $b'$, $c'$, $d'$ to the two conics. Let $S$ be a point of intersection of common tangents $RR'$ and $TT'$.

Then the two conics are in perspective with each other, $s$ being an axis of perspective and the lines $a$, $b$, $c$, $d$ corresponding to the lines $a'$, $b'$, $c'$, $d'$. (Art. 127.)

Hence the points of intersection of $a$, $b$, $c$, $d$ with $RR'$ correspond

$R'B'$, $R'C'$, $R'D'$ and the pencils $(R\,.\,ABCD)$ and $(R'\,.\,A'B'C'D')$ are projective.

to the points of intersection of $a'$, $b'$, $c'$, $d'$ with $RR'$. Hence since $RR'$ is a tangent to both conics the two systems of tangents are projective.

These theorems may also be deduced by conical projection or plane perspective from the corresponding theorems for the circle, which were proved in Art. 84 (11). The theorems also hold when $S$ and $s$ are respectively a common involution point and a common involution chord.

### EXAMPLE.

Through the point of intersection of a pair of common tangents to two conics $C_1$, $C_2$, a straight line is drawn cutting the conics in $Q$, $R$ respectively: $A$, $B$ are fixed points on the conics $C_1$, $C_2$ respectively. Show that the locus of intersection of $AQ$, $BR$ is one or other of two conics.

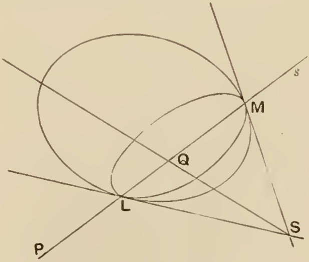
### Conics having Double Contact.

**130.** (*a*) *If two conics have double contact, each may be looked upon as its own perspective, the perspective being harmonic, and the centre and axis of perspective either* (1) *the pole of the chord of contact and the chord of contact, or* (2) *any point on the chord of contact and its polar.*

Case (1) is obvious at once, since the common chord of contact has the same pole with respect to both the conics.

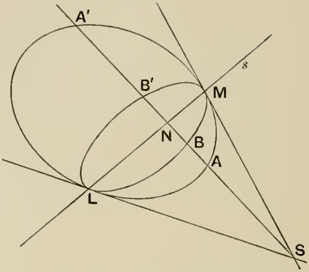
Case (2) follows from the fact that any point on the common chord of contact has the same polar with respect to both conics.

Let the two conics touch at $L$ and $M$ and let $S$ be the common pole of $LM$ ($s$) with respect to the conics. Take $P$ any point on $s$ and $Q$ the harmonic conjugate of $P$ with respect to $L$, $M$. Then $Q$ and $S$ are points on the polar of $P$ with respect to both the conics and therefore $QS$ is the polar of $P$ with respect to both conics.



(*b*) *If two conics have double contact, each may be looked upon as the perspective of the other, the centre and axis of perspective being the pole of the chord of contact and the chord of contact.*

Let the two conics touch at $L$ and $M$ and let $S$ be the pole of $LM$ ($s$). Through $S$ draw a line to meet one conic in $A$ and $A'$, the other in $B$ and $B'$, and the chord of contact $LM$ in $N$.

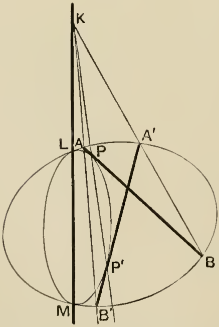


Take $S$ and $s$ as centre and axis of perspective and $(SNAB)$ as the anharmonic ratio of the perspective.

Then two points at $M$ on one conic correspond to two points at $M$ on the other; two points at $L$ on one conic correspond to two points at $L$ on the other; and one point at $A$ on one conic corresponds to one point at $B$ on the other. Hence the two conics are in perspective.

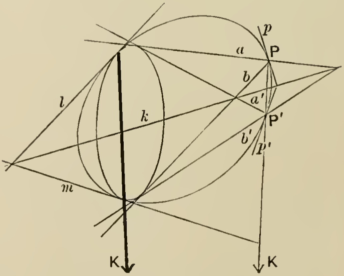
Similarly it may be proved that the conics are in perspective with $M$ and $SL$, or $L$ and $SM$, as centre and axis of perspective.

**131.** *If two conics have double contact at L and M, and tangents at P and P′ to one meet the other in A, B and A′, B′, then a pair of lines AB′, BA′ are concurrent with PP′ at a point on LM.*



Let $PP'$ meet $LM$ in $K$. Then both the conics may be looked

*If two conics have double contact, the common tangents being l and m, and the tangents from points P and P′ on one conic to the other be a, b and a′, b′, then a pair of points ab′, ba′ and the point of intersection of the tangents at P and P′ are collinear with lm.*



Let the connector of $lm$ and $pp'$ be $k$, where $p$ and $p'$ are the upon as in self-perspective with $K$ as centre of perspective and its polar as axis. (Art. 130 ($a$).)

To the tangent at $P$ will correspond the tangent at $P'$. Therefore to $A$ and $B$ will correspond $B'$ and $A'$. Therefore $AB'$ and $BA'$ are collinear with $K$.

*If two conics have double contact and any tangent to one of them at* $P$ *meets the other at* $A$ *and* $B$, *then the ranges described by* $P, A, B$ *on the two conics are projective.*

If in the figure the points $A, P, B$ are kept fixed, and $A'P'B'$ is variable, the ranges described by $A', P', B'$ on the conics are projective with the range described by $K$ on $LM$ and therefore with each other.

The points of contact $L$ and $M$ of the conics are the self-corresponding points of the ranges described by $A'$ and $B'$.

Conversely:

*The lines joining corresponding points of two projective ranges on a conic envelope a conic having double contact with the given conic.*

Construct, Art. 109, the self-corresponding points $L$ and $M$ of the two projective ranges.

tangents at $P$ and $P'$. If $PP'$ meets the common chord of contact at $K$, $K$ is the pole of $k$ with respect to both conics. Take $K$ and $k$ as centre and axis of harmonic perspective.

To the tangent at $P$ corresponds the tangent at $P'$ and to $a$ and $b$ correspond $b'$ and $a'$. Hence $ba'$, $ab'$, $lm$, $pp'$ all lie on $k$.

*If two conics have double contact and the tangents to one from the point of contact of a tangent* $p$ *to the other be* $a$ *and* $b$, *then the systems of tangents* $p$, $a$, $b$ *to the two conics are projective.*

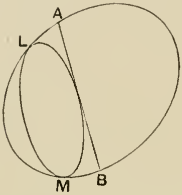
If in the figure the tangents $a, p, b$ are kept fixed, and $a'p'b'$ is variable, the systems of tangents $a'$, $p'$, $b'$ are projective with the pencil described by the line $k$ and therefore with each other.

The tangents $l$ and $m$ at the points of contact of the conics are the self-corresponding tangents of the two projective systems of tangents described by $a'$ and $b'$.

*The locus of the points of intersection of corresponding tangents of two projective systems of tangents to a conic is a conic having double contact with the given conic.*

Construct, Art. 109, the self-corresponding tangents $l$ and $m$ of the two projective systems of tangents.

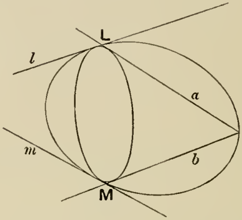
Draw the tangents $l$ and $m$ at $L$ and $M$ and describe a conic to touch $l$ and $m$ at $L$ and $M$ and to touch the line $AB$ joining a pair of corresponding points on the conic. (Art. 113 ($C$).)



By the last Article any tangent to this conic will meet the given conic in a pair of corresponding points of the two projective ranges determined by $L$ and $M$ as self-corresponding points and $A$ and $B$ as a pair of corresponding points.

This conic is therefore the envelope of the chords joining corresponding points of the given projective ranges.

Construct the points of contact $L$ and $M$ of the tangents $l$ and $m$ and describe a conic to touch $l$ and $m$ at $L$ and $M$ and pass through $ab$ the point of intersection of a pair of corresponding tangents. (Art. 113 ($C$).)



By the last Article any pair of tangents from a point on this conic will form a pair of corresponding tangents of the two systems of projective tangents determined by $l$ and $m$ as self-corresponding tangents and $a$ and $b$ as a pair of corresponding tangents.

This conic is therefore the locus of the points of intersection of corresponding tangents of the two projective systems of tangents.

The correlative of Desargues' Theorem for a system of conics with double contact is as follows:

*If a system of conics have double contact at $L$ and $M$, and $E$ be the common pole of $LM$, then the pairs of tangents from any point $P$ to the conics of the system are pairs of conjugate rays of an involution of which $PE$ is a double ray and of which $PL$ and $PM$ are a pair of conjugate rays.*

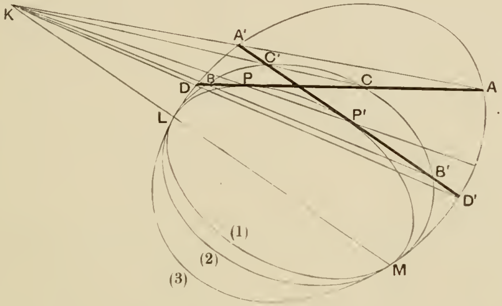
The other double ray of the involution is the tangent at $P$ to the conic of the system which can be described through $P$. (Art. 113 ($C$).)

From Art. 125 the following property of two conics having double contact is obtained:

If two conics touch at $L$ and $M$ and these points are joined to any two points $A$ and $B$ on the one conic by lines, which meet the other conic in $A'$, $B'$, and $A''$, $B''$, respectively, then $A'B''$ and $A''B'$ intersect on the common chord of contact of the conics.

**132.** *Given three conics all touching each other at the same two points, any tangent to one is cut by the other two in four points whose anharmonic ratio is constant.*

($a$) Let the three conics (1), (2), (3) touch at $L$ and $M$, and let the tangents at $P$ and $P'$ to (1) meet (2) and (3) in $A$, $C$, $B$, $D$ and $A'$, $C'$, $B'$, $D'$.



Then $AA'$ and $DD'$ pass through the point $K$ where $PP'$ meets $LM$. Similarly $CC'$ and $BB'$ pass through $K$.

Hence the ranges $ACBD$ and $A'C'B'D'$ are in perspective from $K$ and therefore their anharmonic ratios are equal.

The correlative theorem is :

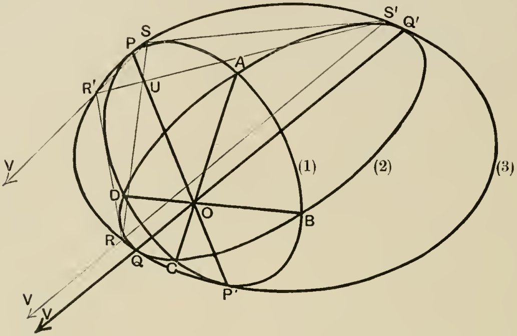
*Given three conics all touching each other at the same two points, the two pairs of tangents from any point on one to the other two form a pencil of constant anharmonic ratio.*

In most theorems concerning two or more conics having double contact at $L$ and $M$, the pair of tangents at $L$ and $M$ may be substituted for one of the conics having double contact. In the correlative theorems the points of contact $L$ and $M$ may themselves be looked upon as a conic of the system having double contact at $L$ and $M$.

($b$) *If two conics* (1) *and* (2) *have each double contact with a third conic* (3), *the chords of contact of* (1) *and* (2) *with* (3) *and two of the chords of intersection of* (1) *and* (2) *meet in a point and form a harmonic pencil.*

Let the points of intersection of the conics (1) and (2) be $A$, $B$, $C$, $D$ and let $PP'$ and $QQ'$ be their chords of contact with (3). Let $PP'$, $QQ'$ meet at $O$.

Since (3) and (1) have double contact at $P$ and $P'$, every point on $PP'$ has the same polar with respect to the two conics. Similarly every point on $QQ'$ has the same polar with respect to (3) and (2). Therefore $O$ the point of intersection of $PP'$, $QQ'$ has the same polar with respect to all three conics. Therefore all the conics are in self-perspective with $O$ as centre and the polar of $O$ as axis of perspective. But the points of intersection of corresponding curves are corresponding points, and, therefore, $BD$ and $AC$ must pass through $O$.



Draw a pair of common tangents to (1) and (2) to meet (3) in $R$, $R'$ and $S$, $S'$. Then (Art. 131) $SR$ and $S'R'$ will meet in a point $U$ on $OP$ and $SR'$ and $S'R$ in a point $V$ on $OQ$.

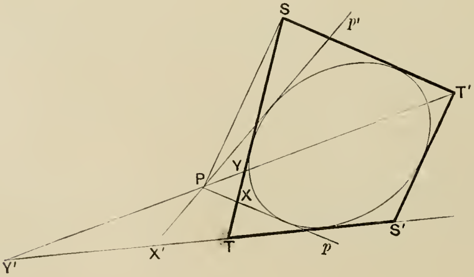
By construction $U$ and $V$ are conjugate points with respect to (3).

But $U$ being on $POP'$ has the same polar with respect to (1) and (3),

and $V$ being on $QOQ'$ has the same polar with respect to (2) and (3).

Hence $U$ and $V$ are common conjugate points with respect to (1) and (2). Hence they are conjugate points with respect to all conics through $A$, $B$, $C$, $D$. Hence they are conjugate points with respect to $AC$ and $BD$. Therefore the pencil $(O.ADUV)$ is harmonic.

(c) *If a quadrangle $STS'T'$ be circumscribed to a conic, and on any tangent $p$ to the conic a point $P$ be taken such that $PS$ is the harmonic conjugate of $p$ with regard to $PT$ and $PT'$, then the locus of $P$ is a conic having double contact with the given conic.*



Let $p'$ be the second tangent from $P$ to the conic. Let $p$ meet $TS$ in $X$ and let $p'$ meet $TS'$ in $X'$. Join $PT'$ to meet $TS$ in $Y$ and $TS'$ in $Y'$.

Then $(P\,.\,TT'XS)=(P\,.\,T'TX'S')$ by the correlative to Desargues' theorem. Hence, since the first of these pencils is harmonic, the second is also harmonic; therefore $(TYXS)$ and $(Y'TX'S')$ are harmonic.

Therefore since $S$ and $T$ are fixed the ranges $Y$ and $X$ are projective,

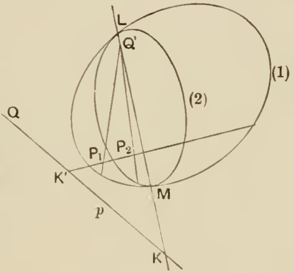
and since $S'$ and $T$ are fixed the ranges $Y'$ and $X'$ are projective.

But since $YY'$ passes through $T'$, the ranges $Y$ and $Y'$ are projective.

Therefore the ranges $X$ and $X'$ are projective and the tangents from $P$ to the conic form two projective systems of tangents. Therefore the locus of $P$ is a conic having double contact with the given conic. (Art. 131.)

(*d*) *To find the locus of common conjugates of points situated on a given straight line with respect to two conics having double contact.*

Let the two conics (1) and (2) have double contact at $L$ and $M$. Consider the locus of common conjugates of points on a line $p$ which meets $LM$ in $K$. Let $P_1$ and $P_2$ be the poles of $p$ with respect to (1) and (2). They lie on the common polar of $K$. Take $Q$ any point on $p$. Its polar with respect to (1) passes through $P_1$ and meets $LM$ in a point $Q'$. Since $Q'$ is on $LM$ its polar with respect to (2) passes through $Q$. Therefore the polar of $Q$ with respect to (2) passes through $Q'$ and is the line $P_2Q'$. Hence $LM$ is part of the locus of common conjugates of points on $p$.



Again, $P_1P_2$ is the polar of $K$ with respect to (1) and (2). Hence the polars of every point of $P_1P_2$ with respect to (1) and (2) pass through $K$.

Hence $P_1P_2$ the polar of $K$ is part of the locus which consists of $LM$ and $P_1P_2$. The above may be stated as follows.

*The polars of any point with respect to two conics which have double contact intersect on their chord of contact.*

(*e*) *To find the locus of the common conjugates of three conics two of which have double contact with the third.*

Let the conics (1) and (3) have double contact at $L$ and $M$.

Let the conics (2) and (3) have double contact at $L'$ and $M'$.

Consider a straight line $p$ which meets $LM$ in $A$ and $L'M'$ in $B$.

Let the common polar of $A$ with respect to (1) and (3) meet $ML$ in $A'$ and $M'L'$ in $A''$.

Let the common polar of $B$ with respect to (2) and (3) meet $ML$ in $B''$ and $M'L'$ in $B'$.

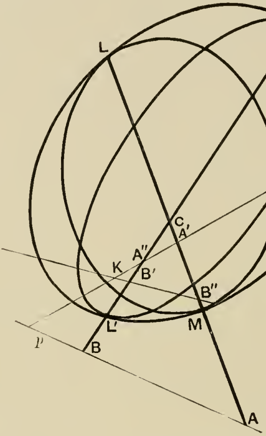
The locus of common conjugates of points on $p$ with respect to (1) and (3) is $LM$ and $A'A''$.

The locus of common conjugates of points on $p$ with respect to (2) and (3) is $L'M'$ and $B'B''$.

These loci intersect in $C(LM . L'M')$, $K(A'A'' . B'B'')$ and $A''$, $B''$. The point $K$ is the pole of $p$ with respect to (3) and is not a point on the locus.

$A''$, $B''$ are common conjugates of $A$ and $B$ and are points on the locus. For different positions of the line $p$, the loci of $A''$ and $B''$ are the lines $LM$ and $L'M'$.

$C$ is a fixed point. It has the same polar with respect to the three conics. The polars of every point on its polar pass through $C$. Hence the polar of $C$ is part of the locus. The locus, therefore, consists of three straight lines, viz. $LM$, $L'M'$ and the common polar of $C$.



## EXAMPLES.

(1) Two fixed conics touch at $A$ and $B$. $LM$ is a chord of one touching the other at $Q$. Prove that the line joining $Q$ to the intersection of $AM$ and $BL$ passes through a fixed point.

Let $LM$ meet $AB$ in $Q'$ and let $AM . BL$ be $P$. Then the conics are in self-perspective with $Q'$ as centre of perspective and its polar $QP$ as axis. In this perspective $MA$ and $BL$ correspond as do the tangents at $A$ and $B$. Therefore $PQ$ must pass through their point of intersection which is a fixed point.

(2) Through a fixed point $O$ is drawn a chord $OPP'$ of a given conic, and on it is taken a point $Q$ such that the range $OPQP'$ has a constant anharmonic ratio. Prove that the locus of $Q$ is part of a conic having double contact with the given conic, and that if $Q'$ be the point in which $OP$ meets the other part of the conic, the product $(OQPP')(OQ'PP')$ is unity. See Art. 118. 3.

(3) A variable tangent to a parabola meets two fixed tangents in $P$, $Q$, and the diameters through $P$, $Q$ meet the curve in $L$, $M$. Determine the envelope of $LM$.

$P$ and $Q$ are two projective ranges on the fixed tangents. $P\infty$, $Q\infty$ are two projective pencils, where $\infty$ is the point at infinity on the axis of the parabola.

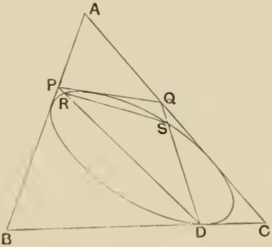
Therefore $L$, $M$ are two projective ranges on the conic. Therefore the envelope of $LM$ is a conic which may be shown to be a parabola.

(4) The triangle $PQR$ is inscribed in a given conic. If the directions of $PQ$ and $PR$ are given, show that $QR$ envelopes a conic having the same asymptotes as the given conic.

By Art. 131 the envelope of $QR$ is a conic having double contact with the given hyperbola. The line $QR$ may be constructed by taking any point $P$ on the hyperbola and drawing lines in fixed directions to meet the curve in $Q$ and $R$. As $P$ moves to infinity in the direction of one asymptote the points $Q$ and $R$ move to infinity in the direction of the other asymptote and at last coincide in a double point which is at infinity. Hence the envelope has double contact with the hyperbola at infinity and has therefore the same asymptotes.

(5) A fixed conic inscribed in the triangle $ABC$ touches the side $BC$ in $D$ and $P$, $Q$ are the points in which any variable tangent to the conic cuts $AB$, $AC$. Prove that if $DP$, $DQ$ cut the conic in $R$, $S$, the envelope of $RS$ is a conic and find where it meets the given conic.

The ranges described by $P$ and $Q$ are projective. Therefore the ranges described by $R$ and $S$ are projective and the envelope of $RS$ is a conic. The self-corresponding points of the ranges coincide at $D$, at which point the conics have four point contact.



(6) If a polygon be inscribed in a conic so that all its sides but one pass through fixed points, then the envelope of that side is a conic having double contact with the given conic. (Art. 131.)

(7) If a polygon be circumscribed to a conic so that all its vertices but one lie on fixed lines, then the locus of that vertex is a conic having double contact with the given conic. (Correlative of Ex. (6).)

(8) Two systems of conics have double contact with a given conic at pairs of given points. Prove that one system of their chords of intersection forms an involution pencil.

(9) Show that, in general, three systems of conics may be described to have double contact with each of two given conics and that the chords of contact of the conics of each system form an involution pencil with a vertex at one of the vertices of the common self-conjugate triangle of the given conics.

(10) The chords of intersection of two conics of the same system, which have double contact with two given conics, pass through a point of intersection of common chords of these conics.

(11) The tangents from any point $P$ to a conic are $p$, $p'$ and its connectors to two fixed points $L$, $M$ on the conic are $l$, $m$. If $(pp'lm)$ is constant, prove that the locus of $P$ is a conic having double contact at $L$ and $M$.

(12) Three conics have double contact at $L$ and $M$. The tangents from any point $P$ on one to the other two are $aa'$, $bb'$ and the connectors of $P$ to $L$, $M$ are $l$, $m$. Prove that $(ablm) = K.(a'b'lm)$, where $K$ is constant.

(13) A conic can be constructed to touch three given straight lines and have double contact with a given conic.

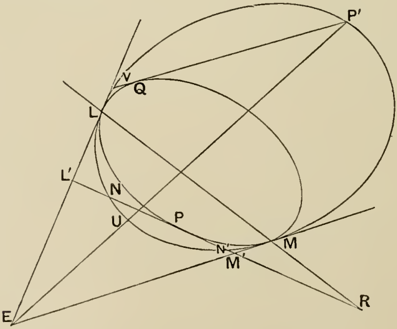
Let the lines $a$, $b$, $c$ meet the conic in $AA'$, $BB'$, $CC'$. Construct the self-corresponding elements $L$ and $M$ of the ranges $ABC$ and $A'B'C'$ on the conic. Then a conic touching the given conic at $L$ and $M$ and the line $a$ will be the required conic. For this conic touches the lines joining each pair of corresponding points of the ranges determined by $L$ and $M$ as self-corresponding points and $A$ and $A'$ as corresponding points.

One conic can be described to touch two given lines at given points and to touch another given line. (By particular case of Brianchon's theorem.) There are, however, four ways of arranging the pairs of points $AA'$, $BB'$, $CC'$ so as to determine different projective ranges on the conic, and there are therefore four solutions of the problem.

(14) The points of contact of the tangents drawn from three points $A$, $B$, $C$ to a conic $S$ are the three pairs of points $A_1A_2$, $B_1B_2$, $C_1C_2$ respectively. Show that the four chords of contact of the four conics that can be drawn through $A$, $B$, $C$ to have double contact with $S$ are the Pascal lines of the four hexagons

$$A_1B_1C_1A_2B_2C_2, \qquad A_1B_1C_2A_2B_2C_1,$$
$$A_1B_2C_1A_2B_1C_2, \qquad A_2B_1C_1A_1B_2C_2.$$

(15) Two conics have double contact at $L$ and $M$ and $E$ is the pole of $LM$. (1) If any tangent to one at $P$ meet the other in $N$ and $N'$ $(LMNN') = (E.LMPN')$, and (2) if a tangent be drawn from any point $P'$ on one to touch the other at $Q$, $(P'.LMEQ)$ is constant.



(1) Let $NPN'$ meet $EL$, $EM$ and $LM$ in $L'$, $M'$, $R$. Then

$$(LMNN') = (L'RNN') \text{ and } (E.LMPN') = (L'M'PN').$$

It is necessary therefore to prove that

$$(L'N'RN) = (L'N'M'P).$$

But $P$ and $R$ are the double points of an involution of which $N$, $N'$; $L'$, $M'$ are two pairs of conjugate points and therefore (Ex. 5, Chapter VIII) this relation holds.

(2) If $P'Q$ and $P'E$ meet the conic, which passes through $P'$, in $V$ and $U$, $(P'.LMEQ) = (LMUV)$ and $(LMUP') = -1$. Therefore $(P'.LMEQ) = -\dfrac{(LMUV)}{(LMUP')}$ $= -(LMP'V)$. This by Art. 131 is constant.

## The Harmonic Locus of Two Conics.

**133.** The following theorems are of importance in connexion with the theory of two conics.

*The envelope of a line, which meets two conics in two pairs of points which are harmonic conjugates, is a conic which touches the eight tangents which can be drawn to the two conics at their points of intersection and has the common self-conjugate triangle of the two conics for a self-conjugate triangle.*

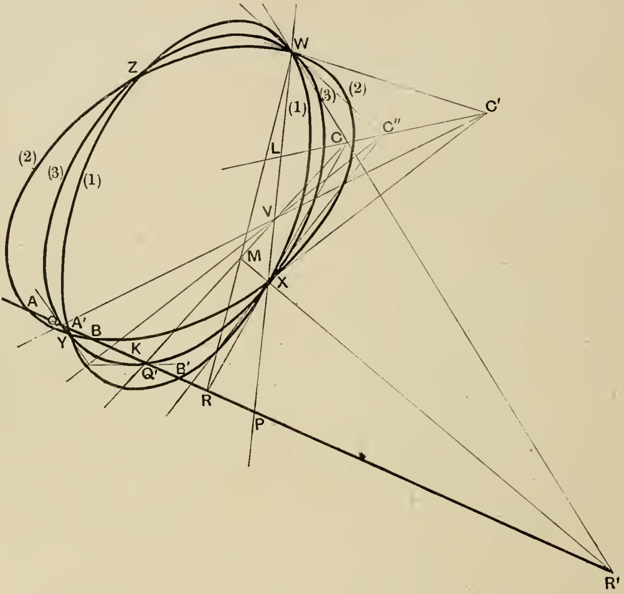
*The locus of a point, the two pairs of tangents from which to two conics are harmonic conjugates, is a conic, which passes through the eight points of contact of the common tangents to the two conics and has the common self-conjugate triangle of the two conics for a self-conjugate triangle.*

Conversely:

*If a conic be drawn to touch the eight tangents which can be drawn to two conics at their four points of intersection, any tangent to this conic is cut by the two conics in two pairs of points which are harmonic conjugates.*

*If a conic be drawn through the eight points of contact of the four common tangents to two conics, the two pairs of tangents which can be drawn to the two conics from any point on this conic are pairs of harmonic conjugates.*

Let the two given conics (1) and (2) intersect in $W$, $X$, $Y$, $Z$. Draw the chord $WX$ and let the tangents to (1) at $W$ and $X$ intersect in $C$ and those to (2) in $C'$. Join $CC'$ meeting $XW$ in $L$. On this line take $C''$ the harmonic conjugate of $L$ with regard to $C'C$. Join $WC''$, $XC''$ and describe a conic through $W$, $X$, $Y$, $Z$ to touch $WC''$ and $XC''$ at $W$ and $X$ respectively (Ex. 5, Art. 125). Call this conic (3). Take any point $V$ on $WX$ and let $CV$ and $C'V$ meet (3) in $Q'$ and $Q$. Join $QQ'$ to meet $WX$ in $P$ and $C''V$ in $K$. Then the range $QQ'KP$ is harmonic and, since the tangents at $X$ and $W$ to (3) meet at $C''$, $C''VK$ is the polar of $P$ with respect to (3). Therefore $WXVP$ is harmonic and $CVQ'$ and $C'VQ$ are the polars of $P$ with regard to (1) and (2) respectively.



Let $QQ'$ meet (2) in $A$ and $B$ and (1) in $A'$, $B'$. Then since the conics (1), (2), (3) pass through four fixed points $W$, $X$, $Y$, $Z$, the pairs of points $AB$, $A'B'$, $QQ'$ are pairs of conjugate points of an involution. Also $Q$ and $Q'$ are harmonic conjugates of $P$ with respect to $AB$ and $A'B'$ respectively. Hence, by Ex. 9 (*b*), Chapter VIII, $A$, $B$ are harmonic conjugates of $A'$, $B'$.

Every chord cut harmonically may thus be constructed by taking different positions of $V$ on the line $WX$, and the locus of $Q$ and $Q'$ is the conic (3) which passes through $W$, $X$, $Y$, $Z$ and touches $C''X$ and $C''W$ at $X$ and $W$.

Consider the tangents $CW$, $CX$ to the conic (1). Let $CX$ and $CW$ meet $QQ'$ in $R$ and $R'$. Since the pencil $(C.\ WXVP)$ is harmonic, $R'X$ and $RW$ will intersect in some point $M$ on $CV$, and, from the harmonic property of the quadrangle $WCXM$, $C$, $M$ will be harmonic conjugates of $Q'$, $V$. Now for different positions of $V$ on $WX$, $C$ is fixed, and $Q'$ describes a conic through $W$ and $X$; therefore $M$ will describe a conic through $W$ and $X$ (Ex. 2, Art. 118). Hence the pencils $WM$ and $XM$ are projective. Therefore the ranges described by $R$ and $R'$ on $CX$ and $CW$ are projective. Therefore $RR'$ will envelope a conic which touches $CW$ and $CX$.

By symmetry the conic also touches the tangents $WC'$ and $XC'$ to (2) and likewise the tangents to (1) and (2) at $Y$ and $Z$. Since the conic touches the tangents at $X$, $Y$, $Z$, $W$ to (1) and (2), it has the same self-conjugate triangle as (1) and (2).

The correlative theorem may be proved in a similar way.

The converse theorems are obviously true as only one conic can be described to touch the eight tangents which can be drawn to two conics at their points of intersection and only one conic can be drawn through the eight points of contact of their common tangents.

*Alternative Proofs.*

($a$) By Projection or Plane Perspective.

If it be assumed that by an imaginary projection $X$ and $W$, two of the points of intersection of the conics, may be projected into the circular points at infinity, the poles $C$ and $C'$ of $XW$ with respect to the two conics become the centres of the circles into which the conics are projected. The envelope touches $WC$, $WC'$, $XC$, $XC'$. Therefore, if $W$ and $X$ are the circular points at infinity, $C$ and $C'$ must become the foci of the envelope. Hence the theorem becomes the following:

*The envelope of a chord which is cut by two circles in pairs of harmonic conjugates is a conic having the centres of the circles for foci.*

This follows from Addendum 14 by the converse of Art. 139 (7).

($b$) It may easily be proved that the locus is of such a nature that two tangents may be drawn to it from any point. Hence there is reason to suppose that it is a conic. (See Art. 142.)

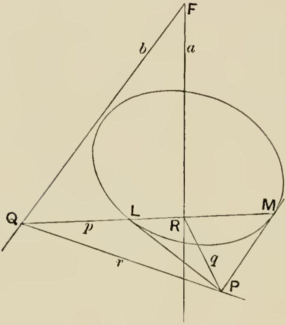
**134.** If in the theorem of Art. 133 a pair of straight lines be substituted for one of the conics and in the correlative one of the conics becomes a pair of points, the following theorems are obtained.

*The envelope of a line, such that its points of intersection with a given conic are harmonic conjugates of its intersections with a pair of given lines, is a conic which touches the pair of given lines.*

*The locus of a point, such that the tangents from it to a given conic are harmonic conjugates of its connectors with a pair of given points, is a conic which passes through the pair of given points.*

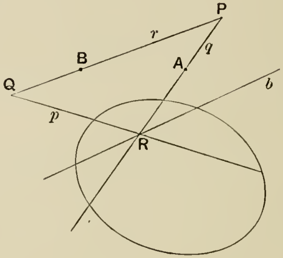
These theorems may be proved independently as follows:

Let $a$ and $b$ be the given lines and $p$ a tangent to the required envelope. Then $pa$, $pb$ ($R$ and $Q$) are conjugate points with respect to the conic. Hence if $P$, the pole of $p$, be joined to $Q$ and $R$ by $r$ and $q$, $pqr$ is a self-conjugate triangle.



Hence $r$ is the polar of $R$, and the range described by $R$ on the fixed line $a$ is projective with the pencil described by $r$ through $A$ the pole of $a$, and therefore with the range described by $Q$ on $b$. Hence the envelope of $p$ is a conic touching $a$ and $b$.

Let $A$ and $B$ be the given points and $P$ a point on the required locus. Then $PA$, $PB$ ($q$ and $r$) are conjugate lines with respect to the conic. Hence, if the polar of $P$ meets $r$ and $q$ in $Q$ and $R$, $PQR$ is a self-conjugate triangle.



Hence $R$ is the pole of $r$, and the pencil formed by $r$, which passes through $B$, is projective with the range described by $R$ on $b$ the polar of $B$, and therefore with the pencil $q$ through $A$. Hence the locus of $P$ is a conic through $A$ and $B$.

The right-hand side of the above may also be stated as follows:

*The locus of the points of intersection of tangents to a given conic from pairs of conjugate points of an involution is a conic, which passes through the double points of the involution.*

*Particular Cases.*

If the lines $a$ and $b$ in the left-hand figure are the connectors of a point $F$ to the circular points at infinity, the chord of intersection $LM$

of $p$ with the conic will subtend a right angle at $F$, which will be a focus of the envelope. Hence the following is obtained.

*The envelope of a chord of a conic which subtends a right angle at a fixed point is a conic having the fixed point for a focus.*

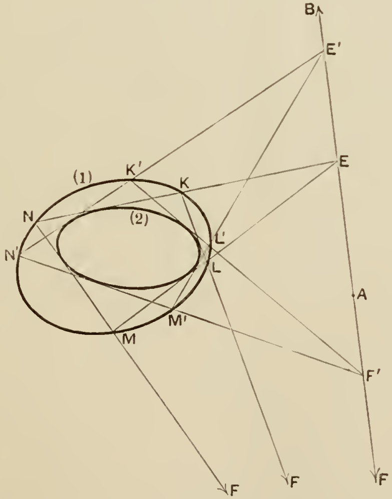
If the points $A$ and $B$ in the right-hand side figure are taken for the circular points at infinity the following is obtained.

*The locus of the points of intersection of tangents to a conic which are at right angles is a circle.*

If $a$ be the polar of $A$, the connectors of $ab$ to $A$ and $B$ are the tangents at $A$ and $B$ to the locus.

From the preceding the following important theorem may be deduced.

*If a simple quadrilateral can be inscribed in one conic and circumscribed to another, an infinite number of quadrilaterals can be similarly constructed.*



Let $KLMN$ be the simple quadrilateral and let the two remaining points of intersection of the sides be $E$ and $F$.

From any point $K'$ on the conic (1) draw a pair of tangents to (2) to meet $EF$ in $E', F'$. Take $A$ and $B$ the common harmonic conjugates of $EF$ and $E'F'$. Then the lines joining $A$, $B$ to $K$, $L$, $M$, $N$, $K'$ are harmonic conjugates of the tangents from these points to (2). Hence this holds for every point on (1). Hence, if $K'F'$ meets (1) in $L'$, the other tangent from $L'$ passes through $E'$. If this tangent meets (1) in $M'$, the other tangent from $M'$ to (2) passes through $F'$. Let $E'K'$ and $F'M'$ meet at $N'$. Then $N'E'$ and $N'F'$ are harmonic conjugates of $N'A$ and $N'B$. Therefore $N'$ is on (1). Hence a quadrangle has been inscribed in (1) and circumscribed to (2).

If one of the quadrilaterals $KLMN$ is a parallelogram, all the other quadrilaterals will be parallelograms.

**135.** (*a*) *If five conics be described to touch a given line* 0 *and four out of five given lines* 1, 2, 3, 4, 5 *and* $Q_1$, $Q_2$, $Q_3$, $Q_4$, $Q_5$ *be the points of contact with* 0 *of the conics which do not touch the lines* 1, 2, 3, 4, 5 *respectively, and* $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ *be the points in which* 1, 2, 3, 4, 5 *meet* 0, *then* $P_1Q_1$, $P_2Q_2$, $P_3Q_3$, $P_4Q_4$, $P_5Q_5$ *are pairs of conjugate points of an involution*[*].

Consider the conic (1) which touches 0, 2, 3, 4, 5.

Let 5 meet 0, 1, 2, 3, 4 in $F_0$, $F_1$, $F_2$, $F_3$, $F_4$ respectively.

Then from the conic (1) $\quad (0\,.\,0234)=(5\,.\,0234)$,

therefore
$$(Q_1P_2P_3P_4)=(F_0F_2F_3F_4)\ \ldots\ldots\ldots\ldots(1),$$

similarly from the conic (2)
$$(Q_2P_1P_3P_4)=(F_0F_1F_3F_4)\ \ldots\ldots\ldots\ldots(2),$$

and from the conic (3)
$$(Q_3P_1P_2P_4)=(F_0F_1F_2F_4)\ \ldots\ldots\ldots\ldots(3).$$

From (1) and (2)
$$\frac{(Q_1P_2P_3P_4)}{(Q_2P_1P_3P_4)}=\frac{(F_0F_2F_3F_4)}{(F_0F_1F_3F_4)},$$

$$\therefore\ (Q_1Q_2P_3P_4)(P_1P_2P_3P_4)=(F_1F_2F_3F_4)\ \ldots\ldots\ldots\ldots(\text{i}).$$

Similarly from (1) and (3)
$$(Q_1Q_3P_2P_4)(P_1P_3P_2P_4)=(F_1F_3F_2F_4)\ \ldots\ldots\ldots\ldots(\text{ii}),$$

and from (2) and (3)
$$(Q_2Q_3P_1P_4)(P_2P_3P_1P_4)=(F_2F_3F_1F_4)\ldots\ldots\ldots\ldots(\text{iii}).$$

But
$$(F_2F_3F_1F_4)=-\frac{(F_1F_3F_2F_4)}{(F_1F_2F_3F_4)}.$$

Therefore dividing (ii) by (i)
$$\frac{(Q_1Q_3P_2P_4)(P_1P_3P_2P_4)}{(Q_1Q_2P_3P_4)(P_1P_2P_3P_4)}=\frac{(F_1F_3F_2F_4)}{(F_1F_2F_3F_4)},$$

and
$$\frac{(Q_1Q_3P_2P_4)}{(Q_1Q_2P_3P_4)}(P_2P_3P_1P_4)=(F_2F_3F_1F_4),$$

$\therefore$ from (iii)
$$(Q_1Q_3P_2P_4)=(Q_1Q_2P_3P_4)(Q_2Q_3P_1P_4),$$

$$\therefore\ \frac{Q_1P_2}{Q_3P_2}\cdot\frac{Q_2P_3}{Q_1P_3}\cdot\frac{Q_3P_1}{Q_2P_1}=1.$$

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[*] I am indebted to Mr S. G. Soal of the East London College for this theorem and the proof.

Hence (Art. 53) $Q_1P_1$, $Q_2P_2$, $Q_3P_3$ are pairs of conjugate points of an involution.

Similarly, $Q_1P_1$, $Q_2P_2$, $Q_4P_4$ are pairs of conjugate points of an involution, which must be the same involution.

Similarly, by considering the lines 3 or 4 in place of 5, it is seen that $Q_1P_1$, $Q_2P_2$, $Q_5P_5$ are pairs of conjugate points of an involution.

Therefore $P_1Q_1$, $P_2Q_2$, $P_3Q_3$, $P_4Q_4$, $P_5Q_5$ are five pairs of conjugate points of an involution.

The correlative theorem is as follows :

*If five conics be described through a given point $O$ and through four out of five given points* 1, 2, 3, 4, 5 *and* $q_1$, $q_2$, $q_3$, $q_4$, $q_5$ *be the tangents at $O$ to the conics which do not pass through* 1, 2, 3, 4, 5 *respectively, and* $p_1$, $p_2$, $p_3$, $p_4$, $p_5$ *be the connectors of $O$ to the points* 1, 2, 3, 4, 5 *respectively, then* $p_1q_1$, $p_2q_2$, $p_3q_3$, $p_4q_4$, $p_5q_5$ *are pairs of conjugate rays of an involution.*

(*b*) *The six polars of a given point with respect to the six conics which pass through five out of six given points touch a conic.*

Consider the polars of the point $O$ with respect to the six conics which pass through five out of the six points 1, 2, 3, 4, 5, 6.

Let $p_1, p_2, p_3, p_4, p_5, p_6$ denote the polars of $O$ with respect to the conics which do not pass through the points 1, 2, 3, 4, 5, 6 respectively.

Consider the intersections of $p_2$, $p_3$, $p_4$, $p_5$ with $p_1$ and $p_6$.

The conics whose polars are $p_1$ and $p_2$ have four points common, viz. the points 3, 4, 5, 6. The polars of $O$ with respect to all conics through 3, 4, 5, 6 will pass through the point of intersection of $p_1$ and $p_2$ which is the common conjugate of $O$ with regard to all such conics (Art. 117). The conic through $O$ and 3, 4, 5, 6 has the tangent at $O$ for the polar of $O$. This tangent consequently passes through the point of intersection of $p_1$ and $p_2$.

Similarly the points $p_1p_3$, $p_1p_4$, $p_1p_5$ are on the tangents at $O$ to the conics which pass through $O$ and through four of the six points other than 1, 3 ; 1, 4 ; 1, 5 respectively.

Similarly the intersections of $p_2$, $p_3$, $p_4$, $p_5$ with $p_6$ lie on the tangents at $O$ to the four conics which pass through $O$ and through the six points other than 6, 2 ; 6, 3 ; 6, 4 ; 6, 5 respectively.

But by the correlative of ($a$), the pencils formed by these two sets of four tangents are projective with the pencil formed by joining $O$ to 2, 3, 4, 5 and therefore with each other. Hence the ranges on $p_1$ and $p_6$ are projective and by the converse of the anharmonic property of tangents to a conic the lines $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, $p_6$ all touch a conic.

The correlative is as follows :

*The poles of a fixed line with respect to six conics which touch five out of six given lines lie on a conic.*

As a particular case, the following theorem is obtained :

*The centres of the six conics which touch five out of six given straight lines lie on a conic.* (Mr H. M. Taylor.)

This may be proved directly as follows.

Denote the sides of the hexagon by $a$, $b$, $c$, $d$, $e$, $f$, the centre of the conic touching $b$, $c$, $d$, $e$, $f$ by $C_a$ and the line joining the middle points of the diagonals of $c$, $d$, $e$, $f$ by $L_{ab}$.

Then (Art. 117. 5) since $L_{ab}$ is the locus of the centres of conics which touch $c$, $d$, $e$, $f$, $C_a$ and $C_b$ lie on $L_{ab}$ and similarly $C_d$, $C_e$ lie on $L_{de}$.

Hence, by the converse of Pascal's Theorem, the condition that $C_a$, $C_b$, $C_c$, $C_d$, $C_e$, $C_f$ should lie on a conic is that $L_{ab} . L_{de}$ ; $L_{bc} . L_{ef}$ ; $L_{cd} . L_{fa}$ should be collinear.

This is the case by Art. 68 ($c$).

## EXAMPLES.

(1) The envelope of a chord which meets a conic in a pair of points, which with its points of intersection with two fixed tangents to the conic form a range of constant anharmonic ratio, is a conic having double contact with the given conic at the points where it touches the fixed tangents.

This is the converse of Art. 132 ($a$) when one of the conics becomes a pair of tangents to the conic.

(2) The locus of the point of intersection of two tangents to a conic, which form a pencil of constant anharmonic ratio with the connectors of the point with two given points on the conic, is a conic having double contact with the conic at the two fixed points.

This is the correlative of the preceding.

(3) The locus of the point of intersection of two tangents to a conic which form a pencil of constant anharmonic ratio with the connectors of the point to two fixed points on any fixed tangent to the conic is a conic having double contact with the given conic.

The tangents determine on the fixed tangents two points which with the two given points on the tangent form a range of constant anharmonic ratio. Therefore the points where the two tangents meet the fixed tangent form two projective ranges. Hence the tangents from these points form two projective systems of tangents, which (Art. 131) intersect on a conic having double contact with the given conic.

(4) The envelope of a line which meets a conic in a pair of points, which form a range of constant anharmonic ratio with its points of intersection with two fixed lines which intersect on the conic, is a conic having double contact with the given conic.

The correlative of the last theorem.

(5) $a$ and $b$ are two lines conjugate with regard to a conic $S$. Prove that, if the points of intersection of a transversal $p$ with $S$ and with $a$ and $b$ form a harmonic range, $p$ must pass through the pole of $a$ or of $b$ with respect to $S$.

(6) If triangles be constructed such that they are self-conjugate with regard to one conic and two sides touch another conic, then the third side will envelope a conic.

(7) If a conic (1) be circumscribed to a quadrangle $ABCD$ and a conic (2) be inscribed in the quadrangle, and a tangent be drawn to (2) to meet (1) in $E$ and $F$, then the other tangents from $E$ and $F$ to (2) will intersect on the line joining $AB.CD$ to $BC.AD$ and their chord of contact will pass through $BD.AC$.

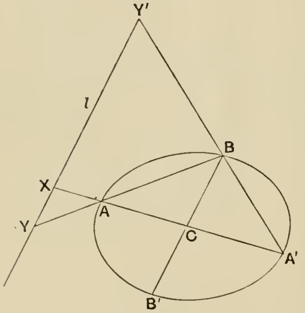
(8) If any tangent to a conic (1) meets a second conic (2), with which it has double contact at $L$ and $M$, in $K$ and $N'$, and $K$ and $N'$ be joined to $S$ any fixed point on $LM$, by lines which meet the conic (2) again in $K'$ and $N$, then the lines $KN$ and $K'N'$ meet $LM$ in two fixed points $A$ and $A'$, which are such that the tangents from $A$ and $A'$ to (2) intersect where the polar of $S$ meets (2).

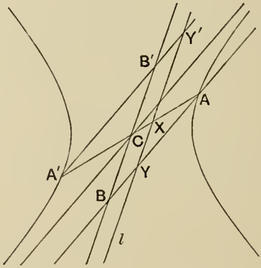
This may be proved by means of the theorems given in Arts. 130–132.

# CHAPTER XXI

## THE CONSTANT OF AN INVOLUTION. FOCAL PROPERTIES AND CONFOCAL CONICS. GEOMETRICAL CORRESPONDENCE. CHARACTERISTICS OF SYSTEMS OF CONICS. STEINER'S AND KIRKMAN'S THEOREMS

**136. To find the constant of the involution determined on any straight line by a central conic.**





(1) Let the line $l$ meet the central conic in imaginary points.

Let $C$ be the centre of the conic. Draw $BCB'$ parallel to $l$, and let the diameter conjugate to $BCB'$ meet the conic in $A$ and $A'$ and $l$ in $X$.

Since $l$ passes through the pole of $AA'$, the pole of $l$ is on $AA'$.

Hence the lines joining $A$ and $A'$ to points on the curve determine conjugate points on $l$ (Art.

Let $C$ be the centre of the conic. Draw $BCB'$ parallel to $l$, and let the diameter conjugate to $BCB'$ meet the conic in $A$ and $A'$ and $l$ in $X$.

Since $l$ passes through the pole of $AA'$, the pole of $l$ is on $AA'$.

Hence the lines joining $A$ and $A'$ to points on the curve determine conjugate points on $l$ (Art.

95 (*j*)), and $X$ is the centre of the involution on $l$.

Join $A$ and $A'$ to $B$ to meet $l$ in $Y$ and $Y'$.

Then

$$XY = XA \cdot \frac{CB}{CA},$$

$$XY' = XA' \cdot \frac{CB}{CA'},$$

$$\therefore\ XY \cdot XY' = -XA \cdot XA' \cdot \frac{CB^2}{CA^2}.$$

95 (*j*)), and $X$ is the centre of the involution on $l$.

Join $A$ and $A'$ to one of the points at infinity on the curve by lines which meet $BCB'$ in $B$ and $B'$ and $l$ in $Y$ and $Y'$.

Then

$$XY = XA \cdot \frac{CB}{CA},$$

$$XY' = XA' \cdot \frac{CB'}{CA'},$$

$$\therefore\ XY \cdot XY' = XA \cdot XA' \cdot \frac{CB^2}{CA^2}.$$

If in the proof for the hyperbola $l$ be moved parallel to itself, $\frac{CB^2}{CA^2}$ is constant, and it follows that $CB$ is the distance from $C$ of the two conjugate points of the involution determined on $BCB'$ by the conic which are equidistant from $C$, and that $-CB^2$ is the constant of this involution.

Hence lines parallel to an asymptote through the ends of a diameter meet its conjugate diameter in the pair of conjugate points of the involution on that diameter, which are equidistant from the centre.

(2) Let the line $l$ meet the central conic in real points.

In the case of the ellipse, and in the case of the hyperbola, when the line meets the same branch of the hyperbola, the proof given in (1) holds.

Thus for the ellipse

$$XY \cdot XY' = -XA \cdot XA' \cdot \frac{CB^2}{CA^2}.$$

But $XA \cdot XA'$ is negative in this case.

Hence there are two double points $P$ and $P'$ of the involution, which are on the curve, such that

$$XP^2 = -XA \cdot XA' \cdot \frac{CB^2}{CA^2}.$$

[Cf. Art. 104 (*a*) 2.]

Thus for the hyperbola

$$XY \cdot XY' = XA \cdot XA' \cdot \frac{CB^2}{CA^2}.$$

But $XA \cdot XA'$ is positive in this case.

Hence there are two double points $P$ and $P'$ of the involution, which are on the curve, such that

$$XP^2 = XA \cdot XA' \cdot \frac{CB^2}{CA^2}.$$

In the preceding (1) and (2), if $K$ be the constant of the involution determined on $BCB'$, then for both the ellipse and hyperbola

$$XY \, . \, XY' = -XA \, . \, XA' \, . \, \frac{K}{CA^2}$$

$$= \left\{1 - \frac{XC^2}{CA^2}\right\} K.$$

$$\therefore \quad \frac{XY \, . \, XY'}{K} = 1 - \frac{XC^2}{CA^2}.$$

Hence, if $P_1$, $P_2$ are a pair of conjugate points of an involution, $X$ its centre, $B_1$, $B_2$ and $A_1$, $A_2$ pairs of conjugate points on the diameter parallel to $P_1P_2$ and on its conjugate, and $C$ the centre of the curve,

$$\frac{XP_1 \, . \, XP_2}{CB_1 \, . \, CB_2} + \frac{CX^2}{CA_1 \, . \, CA_2} = 1.$$

If $P$ be a double point of the involution, for the ellipse

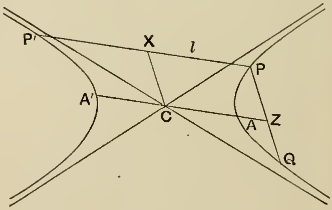
$$\frac{XP^2}{CB^2} + \frac{CX^2}{CA^2} = 1$$

and for the hyperbola

$$-\frac{XP^2}{CB^2} + \frac{CX^2}{CA^2} = 1.$$

The first of these results has been obtained from Carnot's Theorem. (Art. 104 (*a*) (2).)

(3) Let the line $l$ meet different branches of the hyperbola in $P$ and $P'$.



If $X$ be the middle point of $PP'$, $XP^2$ is the constant of the involution on $PP'$. Draw $ACA'$ the diameter parallel to $PP'$, and $PZQ$ parallel to $CX$ to meet $ACA'$ in $Z$.

Then, since $PQ$ and $PP'$ are parallel to conjugate diameters,

$$\frac{CZ^2}{CA^2} - \frac{PZ^2}{CB^2} = 1,$$

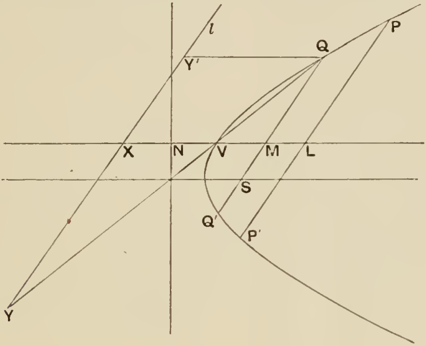
where $-CB^2$ is the constant of the involution on $CX$.

Therefore $XP^2 = CZ^2 = \frac{CA^2}{CB^2}(CB^2 + CX^2)$. This is the constant of the involution on $PP'$.

**137. To find the constant of an involution determined on any straight line by a parabola.**

(1) Let the line $l$ meet the parabola in imaginary points. Take $L$ its pole and draw a chord $PP'$ of the parabola through $L$ parallel to $l$.

Draw a parallel chord through the focus $S$ to meet the parabola in $Q$ and $Q'$. Let $M$ be the middle point of $QQ'$ and let $LM$, which is a diameter of the parabola, meet the curve in $V$, $l$ in $X$, and the directrix in $N$. Let $\infty$ denote the point at infinity on the parabola.



Then $V\infty$ is a chord of the parabola through the pole of $l$. Hence, if $Q$ be joined to $V$ and $\infty$ by lines which meet $l$ in $Y$ and $Y'$, these points are conjugate points of the involution on $l$ of which $X$ is the centre.

Then $$XY = XV \cdot \frac{MQ}{MV} \text{ and } XY' = MQ.$$

$$\therefore\ XY \cdot XY' = XV \cdot \frac{QM^2}{MV}.$$

But since the tangents at $Q$ and $Q'$ intersect at right angles at $N$, a circle with centre $M$ may be described to pass through $Q$, $N$, $Q'$. Therefore $QM = MN$ and, since $MV = VN$,

$$\frac{QM^2}{MV} = \frac{MN^2}{MV} = 4 \cdot VN = 4 \cdot SV.$$

Therefore $$XY \cdot XY' = -\,4 \cdot SV \cdot VX.$$

The negative sign is taken since $XY$ and $XY'$ are drawn in opposite directions.

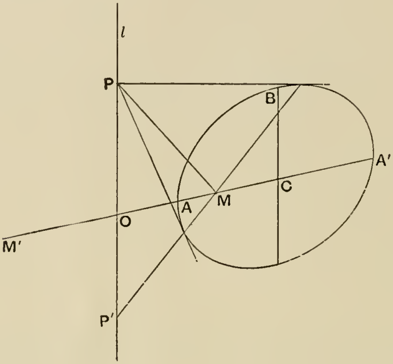
If $P$ and $P'$ are joined to the point at infinity on the curve, a pair of conjugate points are found on $l$, the product of whose distances from $X$ is $-PL^2$.

Therefore $$PL^2 = 4 \cdot SV \cdot VL.$$

(2) If $l$ meets the curve in real points, $P$ and $P'$, the constant of the involution is $PL^2$ which equals $4 . SV . VL$. (Art. 104 ($a$) (3).)

### 138. Gaskin's Theorem.

*The circle, which circumscribes any self-conjugate triangle of a conic, is cut orthogonally by the director circle of the conic.*



Let $PP'M$ be any self-conjugate triangle and let $PP'$ be $l$. Join $M$ to the centre of the conic to meet $l$ in $O$, the conic in $A$, and the circumscribed circle of $PP'M$ in $M'$. Let $CB$ be the diameter conjugate to $CA$.

Then $CM . CM' = CM(CO + OM') = CM\left(CO + \frac{OP . OP'}{OM}\right)$,

since $OP . OP' = OM . OM'$,

$$= CM\left(CO - \frac{CB^2}{CA^2} \cdot \frac{OM . OC}{OM}\right)$$

(by Art. 136), since $OA . OA' = OM . OC$,

$$= CM . CO\left(1 + \frac{CB^2}{CA^2}\right)$$

$$= \frac{CM . CO}{CA^2}(CA^2 + CB^2)$$

$= CA^2 + CB^2$ since $O$ and $M$ are conjugate points

$=$ sum of the squares of the semi-axes. (Art. 139.)

## EXAMPLES.

(1) Show that the circumcircle of a triangle self-conjugate with respect to a rectangular hyperbola passes through the centre of the hyperbola.

The director circle of the rectangular hyperbola is the centre.

(2) The centre of the circumcircle of a triangle self-conjugate with regard to a parabola lies on the directrix of the parabola.

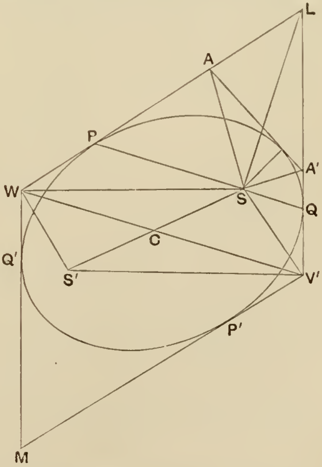
(3) The feet of the perpendiculars drawn from any point on the director circle to the polar of that point and to the three sides of a self-conjugate triangle are concyclic.

Let $P$ be a point on the director circle, $p$ its polar, and $ABC$ a self-conjugate triangle. Let $p$ meet the sides of $ABC$ in $A'$, $B'$, $C'$. Then $PA$, $PA'$; $PB$, $PB'$; $PC$, $PC'$ are pairs of conjugate rays of the involution determined by the conic at $P$, of which the tangents from $P$ are the double rays. Since these are at right angles, the angles $BPC$ and $B'PC'$ are equal. Hence, Addendum 15, the feet of the perpendiculars from $P$ on $p$ and on the sides of $ABC$ are concyclic.

(4) If any two conics are rotated in any manner round their centres the locus of the centre of the circumcircle of their common self-conjugate triangle is the radical axis of their director circles.

### Focal Properties of Conics.

**139.** In Chapter XIII, Arts. 96 and 97, some theorems connected with the foci of a conic were given. In this article certain metrical properties of conics connected with their foci will be proved.



If $LW$, $V'M$; $LV'$, $WM$ be any two pairs of parallel tangents to a conic, and $AA'$ any other tangent, which meets $WL$ and $LV'$ in $A$ and $A'$, then $A$, $A'$; $W$, $\infty$; $\infty$, $V'$ determine on $LW$ and $LV'$ two projective ranges, and any other tangent to the conic meets these lines in pairs of corresponding points of these ranges. (Art. 93.)

By Art. 39 construct points $S$ and $S'$ such that corresponding points of the ranges subtend a constant angle $a$ at $S$, and a constant angle $a'$ at $S'$. Then $S$ and $S'$ are the foci of the conic and are the same points for all pairs of tangents $LW$ and $LV'$. (Art. 98, Example (8).)

From the construction for these points

$$SW \,.\, SV' = WA \,.\, V'A' = \text{a constant},$$

and the angles $AWS$ and $A'V'S$ are equal.

Let $P$, $Q$, $P'$, $Q'$ be the points of contact of $WL$, $LV'$, $V'M$ and $MW$, and $C$ the middle point of $WV'$, which is also the middle point of $SS'$, since $SV'S'W$ is a parallelogram. Then the angles $PSL$ and $LSQ$ are equal to each other and to the angle $ASA'$.

(1) *The tangents from any point to a conic are equally inclined to the focal distances of the point.*

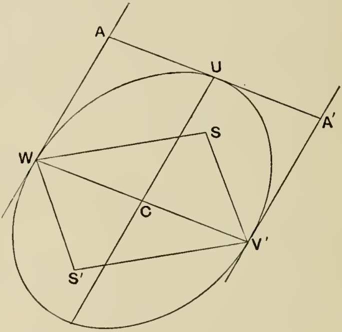
The angle $PWS$ equals the angle $SV'Q$ (Art. 39) and therefore equals the angle $S'WQ'$.

(2) *The tangent at any point to a conic is equally inclined to the focal distances of the point.*

If in (1) the point $W$ is on the conic, the theorem follows at once.

(3) *If a pair of parallel tangents to a conic meet the tangents from a point $W$ in $A$ and $B$, then $WA \,.\, WB$ equals the product of the focal distances of $W$.*

If a tangent $BB'$ parallel to $AA'$ be drawn to meet $WM$ and $MV'$ in $B$ and $B'$, then $A'V' = WB$.



If the tangents $LW$ and $LV'$ are parallel, the points $P$, $W$, $Q'$ and the points $Q$, $V'$, $P'$ coincide in two points $W$ and $V'$ on the conic, which are the ends of a diameter. If $AUA'$ be drawn parallel to $WV'$, it will touch the conic at a point $U$ such that $CU$ is the diameter conjugate to $WCV'$.

(4) If $CW$ and $CU$ are at right angles, they will be the axes of the conic and $CU$ will pass through $S$ and $S'$ (Art. 96). $SW$ will likewise equal $SV'$.

Then for an ellipse, since $WA \,.\, V'A'$ equal $WS \,.\, V'S$,

$$CU^2 = SW^2 = WC^2 + CS^2,$$

$$\therefore\ CS^2 = CU^2 - WC^2 = a^2 - b^2,$$

where $a$ and $b$ are the semi-axes of the conic.

Similarly it may be proved for a hyperbola that $CS^2 = a^2 + b^2$. (Cf. Art. 96.)

In the more general case

(5) $SW \,.\, SV' = WA \,.\, VA' = CU^2$

$= $ square of the semi-diameter conjugate to $CW$.

(Cf. Art. 102, Example (7).)

(6) *The sum of the squares of a pair of conjugate diameters is constant.*

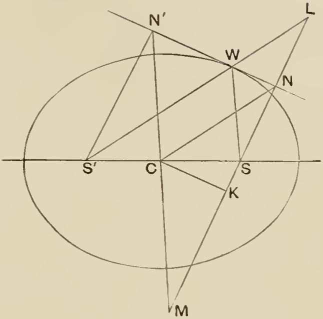
In the figure
$$\begin{aligned}(SW + SV')^2 &= WS^2 + SV'^2 + 2 \,.\, WS \,.\, SV' \\ &= 2 \,.\, SC^2 + 2 \,.\, CW^2 + 2 \,.\, CU^2 \\ &= 2\,(a^2 - b^2) + 2\,(CW^2 + CU^2).\end{aligned}$$

But, Art. 96 ($b$), $SW + SV' = 2a$,

$$\begin{aligned}\therefore\ CW^2 + CU^2 &= \tfrac{1}{2}\,(SW + SV')^2 - (a^2 - b^2) \\ &= a^2 + b^2.\end{aligned}$$

The corresponding theorem for a hyperbola may be proved in a similar way.

(7) *The locus of the foot of a perpendicular from a focus of an ellipse on a variable tangent is a circle concentric with the ellipse.*



Let $SN$ be the perpendicular from $S$ a focus of an ellipse on the tangent at $W$. Produce $SN$ to meet $S'W$ at $L$. Then, since $S'W$ and $SW$ make equal angles with the tangent at $W$, the triangles $SWN$ and $LWN$ are equal. Therefore $SN$ equals $NL$, and $SW$ equals $WL$. Therefore

$$S'L = S'W + WS = 2a.$$

Join $C$ the centre of the conic to $N$. Then $CN$ is parallel to $S'L$. Therefore $CN = \frac{1}{2}S'L = a$. Hence the locus of $N$ is a circle concentric with the ellipse.

The theorem for the hyperbola may be proved in a similar manner.

(8) If $N'$ be the foot of the perpendicular from $S'$, the second focus, on the tangent at $W$ and $CN'$ meets $NS$ at $M$, then $CM$ and $CN$ are equally inclined to the perpendicular $CK$ from $C$ on $NS$, and $MS = S'N'$.

Therefore $SN . S'N' = SM . SN = (NK + SK)(NK - SK) = NK^2 - SN^2$

$$= CN^2 - CS^2 = a^2 - (a^2 - b^2) = b^2.$$

(9) *The area of the parallelogram formed by a pair of tangents parallel to a pair of conjugate diameters is constant.*

Let $CR$ be the perpendicular from $C$ on the tangent at $W$. Then, from the similar triangles $SWN$ and $S'WN'$,

$$\frac{SN}{SW} = \frac{S'N'}{S'W} = \frac{\sqrt{SN . S'N'}}{\sqrt{SW . S'W}} = \frac{b}{CU} \text{ by (8) and (5)}$$

$$= \frac{SN + S'N'}{SW + S'W} = \frac{CR}{a}.$$

Therefore $CU . CR = a . b$.

## Confocal Conics.

**140.** *Conics, which have the same pair of points for their real foci, are termed confocal conics.*

*All conics of a confocal system are inscribed in the same quadrilateral.*

By Art. 101 the tangents—imaginary—from a focus of a conic to the conic pass through the circular points at infinity. Hence the connectors of these points to the two foci form a circumscribed quadrilateral for all conics having the same foci, *i.e.* for all confocal conics. One pair of opposite vertices of the quadrilateral are the two real foci, and another pair are the circular points at infinity. The third pair of vertices are a pair of imaginary foci of the conics, which are situated on the minor or conjugate axis. The common self-conjugate triangle of the conics consists of the two axes of the conics and the line at infinity. The vertices of this triangle are the common centre of the conics and the points at infinity on the axes.

Since confocal conics are inscribed in a quadrilateral one confocal of a given system can be drawn to touch a given straight line, and two confocals can be drawn through any given point (Art. 113).

*Every pair of confocal conics, which intersect in a real point, intersect at right angles.*

Consider the involution pencil determined at any point $P$ by a system of confocal conics and their common circumscribed quadrilateral. Let $S$ and $S'$ be the foci of the conics and $\Omega$ and $\Omega'$ the circular points at infinity. Since $P\Omega$ and $P\Omega'$ are conjugate rays of the involution pencil, the double rays of the pencil, which are harmonic conjugates of these lines, are at right angles. But these double rays are the tangents to the two conics of the system, which pass through $P$. Hence these conics, which are the two conics of the system which pass through $P$, intersect orthogonally.

*The pairs of tangents, which can be drawn from any point to the conics of a confocal system, are equally inclined to the focal distances of the point.*

Since the double rays of the involution pencil determined at any point $P$ by the confocal system are at right angles, the pairs of conjugate rays are equally inclined to the double rays (Art. 16). Since the connectors of $P$ to the foci are a pair of conjugate rays of this involution the result follows at once. The theorem also follows from Art. 139 (1).

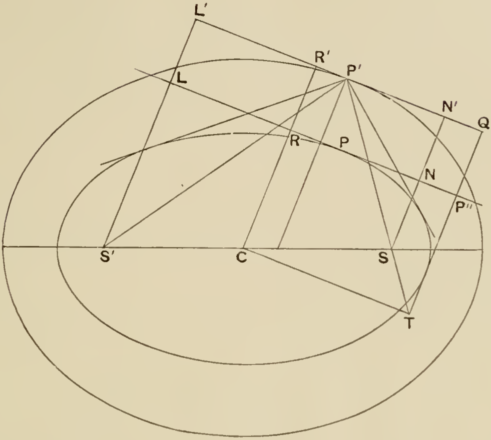
*The locus of the pole of a given straight line, with respect to a system of confocal conics, is the normal at its point of contact to the conic of the system which touches the given straight line.*

Since the confocal system is inscribed in a quadrilateral the locus of the pole of a given straight line $p$ with respect to the conics of the system is a straight line $p'$ (Art. 117 (1)). This line passes through the point of contact of the conic of the system, which touches the line $p$. The lines $p$ and $p'$ are common conjugates with respect to all conics of the system and are therefore harmonic conjugates of all pairs of tangents from $pp'$ to conics of the system. They are therefore at right angles and are the double rays of the involution determined by the conics at $pp'$. Hence they are the tangents to the two conics of the system, which intersect orthogonally at $pp'$.

*The envelope of the polars of a given point, with respect to a system of confocal conics, is a parabola, which touches the axes of the conics.*

Since the confocal system is inscribed in a quadrilateral, the envelope of the polar of a given point $P$ is a conic (Art. 117 (2)). This conic touches the sides of the common self-conjugate triangle of the conics and is therefore a parabola, which touches the axes of the conics. It also touches the tangents $p$ and $p'$ to the conics of the system through $P$. Since $p$ and $p'$ are at right angles, as are also the axes of the conics, the directrix of the parabola is the line $CP$, where $C$ is the centre of the conics. If $p$ and $p'$ meet the axes of the conics in $A_1$, $A_1'$ and $B_1$, $B_1'$, the circles described on $A_1B_1$ and $A_1'B_1'$ as diameters intersect at $C$ and in $F$ the focus of the parabola.

*The difference of the squares of the perpendiculars from the centre on parallel tangents to two confocal conics is constant.*



Let the perpendiculars from the centre $C$ and foci $S$ and $S'$ on two parallel tangents at $P$ and $P'$ meet the tangents in $R$, $N$, $L$ and $R'$, $N'$, $L'$ respectively.

Then

$$CR'^2 - CR^2 = \left(\frac{S'L' + SN'}{2}\right)^2 - \left(\frac{S'L + SN}{2}\right)^2$$

$$= \left(\frac{S'L' - SN'}{2}\right)^2 - \left(\frac{S'L - SN}{2}\right)^2 + S'L' . SN' - S'L . SN$$

$$= \left(\frac{S'L' - SN'}{2}\right)^2 - \left(\frac{S'L' - SN'}{2}\right)^2 + b'^2 - b^2,$$

where $b$ and $b'$ are the semi-minor axes (Art. 139 (8)),

$$= b'^2 - b^2.$$

*The locus of the point of intersection of orthogonal tangents to two confocal conics is a circle which has the same centre as the confocal conics.*

Let $TQ$ be a tangent to one confocal, which meets the tangent at $P'$ to the other orthogonally at $Q$, and let $T$ be the foot of the perpendicular from $C$ on this tangent.

Let $QT$ meet the tangent at $P$ in $P''$.

Then

$$CQ^2 = CR'^2 + CT^2 = CR^2 + b'^2 - b^2 + CT^2 = CP''^2 + b'^2 - b' = a^2 + b^2 + b'^2 - b^2 = a^2 + b'^2.$$

Hence the locus of $Q$ is a circle concentric with the conics.

## EXAMPLES.

(1) $C$ is the centre of two confocal ellipses, $S$ a focus and $A$ and $A'$ the ends of their major axes. Prove that the lines perpendicular to $AA'$ drawn through points $P$, such that $CP.CS = CA.CA'$, are the real chords of intersection of the ellipses.

(2) The locus of the common conjugates, with respect to two confocal ellipses, centre $C$, focus $S$ and ends of the major axes $A$ and $A'$, of points on a line passing through a point $P$ at a distance $CP$ from the centre and parallel to the minor axis, is a straight line parallel to the minor axis, which meets the major axis at a point $P'$ such that $CP.CP' = \frac{CA^2.CA'^2}{CS^2}$ (Art. 125).

(3) If $P$ and $P'$ be two points collinear with a focus on two confocal ellipses centre $C$, foci $S$ and $S'$ and ends of major axes $A$ and $A'$, and these points be joined to $A$ or $A'$, the locus of the point of intersection of $PA$ and $P'A'$ is two straight lines perpendicular to the major axis and passing through points on it at distances $\pm\frac{CA.CA'}{CS}$ from the centre.

(4) Prove that the double rays of the pencils of parallel lines in (2) are the common chords of the two ellipses.

### One to one correspondence.

**141.** One to one correspondence may be defined as follows. *When two groups of geometrical forms (points, straight lines, curves) are so related, that any element of one group determines one and only one element of the other group, and the latter element determines the first element uniquely, the elements of the groups are said to have one to one correspondence.*

The following examples illustrate this principle.

(1) A ray $OQR$ passing through a fixed point $O$ determines points $Q$ on a fixed line $a$, and $R$ on another fixed line $b$, and these points determine rays $AQ$, $BR$ at fixed points $A$, $B$. By construction there is one to one correspondence between $AQ$ and $BR$.

(2) A point $P$ on a conic determines rays $AP$, $BP$ at fixed points $A$ and $B$ on the conic. The construction shows that there is a one to one correspondence between the rays $AP$, $BP$.

(3) A range of points $A$, $B$, $C$, ... at distances $OA$, $OB$, $OC$, ... from a fixed point $O$ on their base, and a second range of points $A'$, $B'$, $C'$, ... on the same base at distances from $O$ such that $OA' = 2.OA$, $OB' = 2.OB$, $OC' = 2.OC$, ... have one to one correspondence.

(4) A pencil of rays $a$, $b$, $c$, ... through a fixed point $S$, which make angles $\widehat{oa}$, $\widehat{ob}$, $\widehat{oc}$, ... with a fixed ray $o$ through $S$, and a second pencil of rays $a'$, $b'$, $c'$, ... through $S$, such that the angles $\widehat{oa'}$, $\widehat{ob'}$, $\widehat{oc'}$, ...

are double of the angles $\widehat{oa}, \widehat{ob}, \widehat{oc}, \ldots$, have not one to one correspondence. For, if $\widehat{oa} = \alpha$ and $\widehat{ob} = \alpha + \frac{\pi}{2}$, the rays $a'$ and $b'$ are given by $\widehat{oa'} = 2\alpha$ and $\widehat{ob'} = 2\alpha + \pi$. Hence the rays $a'$ and $b'$ coincide, and one ray of the second pencil corresponds to more than one ray of the first.

In Chapter VI, Art. 44, it was proved that, if there is an algebraic one to one correspondence between the points of two ranges, these ranges are projective. The following statement summarises the circumstances in which it is safe to infer that a one and one correspondence is algebraic.

*If in a one to one correspondence, as now defined, a point of one range—or a ray of one pencil—is obtained from the corresponding element of the other range—or pencil—by a geometrical construction involving*

(1) *The determination of straight lines, as the connectors of points or by means of the construction of given angles, and*

*The determination of points, as the intersections of straight lines or by the measurement of distances along straight lines,*

(2) *The construction of conics by points or tangents,*

(3) *The determination of points on conics as their points of intersection with other conics or straight lines, and*

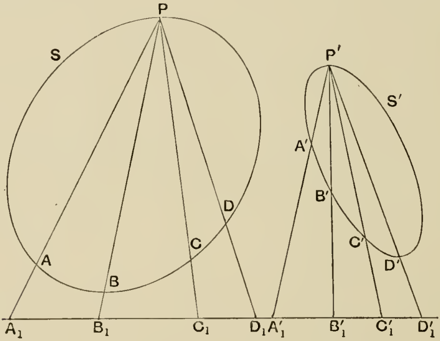
*The determination of tangents to conics, as common tangents to conics, or as tangents from points to conics,*

*then the one to one correspondence is algebraic.*

This theorem or principle depends on the fact that a conic is an algebraic curve and its truth will be realised at once by the student who is acquainted with the elements of coordinate geometry. The degree of a plane algebraic curve is determined by the number of points real, coincident, or imaginary in which a line meets the curve. In Art. 95 ($k$) it was proved that a conic determines on every straight line in its plane a definite involution the double points of which are two points real, coincident, or imaginary. In Art. 101 it was shown, that these points may be regarded as the points of intersection of the line with the conic. Hence a conic is an algebraic curve of the second degree. The class of a curve is defined by the number of tangents to the curve, which may be drawn to it from any point in its plane. In Art. 93 it was shown that a conic is a curve of the second class. Conversely it may be proved that all algebraic curves of the second degree or of the second class are conics. The correspondence here defined will be termed *an algebraic one to one correspondence* or shortly when there is no fear of confusion *a one to one correspondence.* In this case Art. 44 applies and the ranges and pencils are projective. Although the point does not at present arise the theorem now enunciated may be extended by substituting "any algebraic curve" for "conic" in the statement of it.

The theorem that two ranges on linear bases between which there is an algebraic one to one correspondence are projective may be extended to ranges on conics between which there is an algebraic one to one correspondence. The proof excludes the case of ranges obtained by measuring lengths along the curve. Such ranges have not generally a one to one correspondence.

*If there is an algebraic one to one correspondence between two ranges on the same or on two different conics, the ranges are projective.*



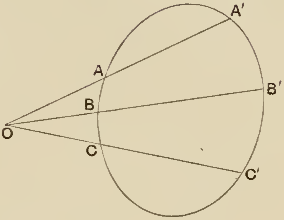
Let $A$, $B$, $C$, $D$ and $A'$, $B'$, $C'$, $D'$ be corresponding points of ranges situated on conics $S$ and $S'$. Join $A$, $B$, $C$, $D$ to any point $P$ on $S$ and $A'$, $B'$, $C'$, $D'$ to any point $P'$ on $S'$. Let the pencils so formed be cut by any line $s$ in $A_1$, $B_1$, $C_1$, $D_1$ and $A_1'$, $B_1'$, $C_1'$, $D_1'$ respectively. Since there is one and one algebraic correspondence between $ABCD$ and $A'B'C'D'$, there is a similar correspondence between $A_1$, $B_1$, $C_1$, $D_1$ and $A_1'$, $B_1'$, $C_1'$, $D_1'$. Hence (Art. 44) these ranges are projective. Therefore the ranges $A$, $B$, $C$, $D$ and $A'$, $B'$, $C'$, $D'$ are also projective.

If the corresponding elements of two ranges, or pencils, having algebraic one to one correspondence mutually correspond, it follows from Art. 51 that they form an involution.

The following are examples of the application of the method of one to one correspondence.

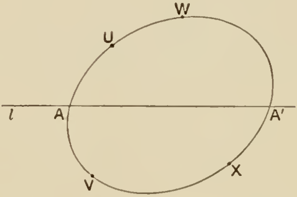
(1) *Involution Property of the Conic.*

Through any point $O$ draw two fixed chords to meet the conic in $A$, $A'$ and $B$, $B'$. These chords may be regarded as determining $O$. Draw $OCC'$ any other chord which may be considered as a variable chord. When $C$ is given $C'$ is uniquely determined, and when $C'$ is given $C$ is uniquely determined, hence the ranges described by $C$ and $C'$ are projective. Since $C$ corresponds to $C'$ and $C'$ to $C$, they form an involution.



(2) *Desargues' Theorem.*

Let $U$, $V$, $W$, $X$ be the four fixed points, through which the system of conics are described, and $l$ the given transversal. On $l$ take any point $A$. Describe a conic through $U$, $V$, $W$, $X$ and $A$. Only one such conic can be described. It will meet $l$ in a point $A'$. If a conic is described through $A'$, $U$, $V$, $X$, $W$ it will pass through $A$. Hence $A$ determines $A'$ uniquely and $A'$ determines $A$ uniquely. Therefore the ranges described by $A$ and $A'$ are projective. Also $A$ and $A'$ mutually correspond and therefore the ranges described by $A$ and $A'$ form an involution



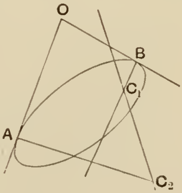
(3) *If a system of conics is described to touch two fixed lines at fixed points $A$, $B$, the envelope of the line joining the corresponding centres of curvature at $A$ and $B$ is a parabola.*

The centres of curvature $C_1$ and $C_2$ are situated on the respective normals at $A$ and $B$. For each conic they are uniquely determined and therefore form two projective ranges on the normals.

Therefore $C_1C_2$ envelopes a conic.

In the limiting case when the conic becomes the straight lines $OA$, $OB$, $C_1$ and $C_2$ are both at infinity.

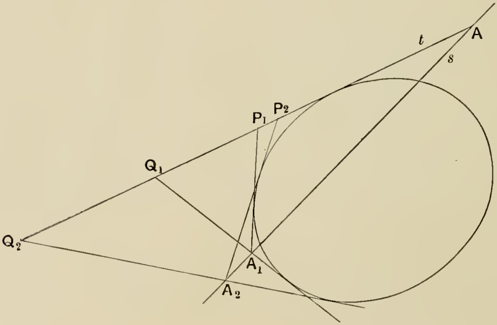
In this case $C_1C_2$ becomes the line at infinity, and therefore the envelope touches the line at infinity and is a parabola.



### 142. Determination of the nature of a locus from its points of intersection with a straight line.

In the last article it was shown that, if the construction of corresponding elements of ranges or pencils, which have one to one correspondence, depends on certain geometrical operations, it is safe to assume that this one to one correspondence is algebraic. If the construction of points on a locus, or of tangents to an envelope, depends on the same operations it follows that the locus or envelope must be an algebraic curve. In this case the degree of the locus may be inferred from the number of points in which any straight line, or a particular straight line, meets the locus, and the class of an envelope may be found from the number of tangents, which may be drawn to it from any point or from a particular point. In the use of this method care should be exercised as geometrically it is often difficult to detect the imaginary points of intersection of a straight line with a locus or the imaginary tangents from particular points. The following, which are generalisations of theorems already proved, are given as illustrations of the method.

(1) *To find the nature of the locus of the points of intersection of tangents to a given conic from corresponding points of two superposed projective ranges.*



Let $t$ be any tangent to the conic. It will meet the base $s$ on which the ranges are situated in some point $A$. The point $A$ may be looked upon as a point of either of the projective ranges and, according to the range to which it is regarded as belonging, it will correspond to a point $A_1$ or a point $A_2$ on $s$. The points $A_1$ and $A_2$ will only coincide, if the ranges form an involution (Art. 51). The tangents from $A_1$ will meet $t$ in two points of the locus $P_1$, $Q_1$. The tangents from $A_2$ will meet $t$ in two other points of the locus $P_2$, $Q_2$. Hence the locus meets $t$ in four different points $P_1$, $Q_1$, $P_2$, $Q_2$ and is therefore of the fourth degree.

This result should be compared with (i) Art. 75, converse, and Art. 95 (*c*), with (ii) Art. 131, converse, and with (iii) Art. 134, which are particular cases. It is thus seen that

*The locus of the points of intersection of tangents to a conic from pairs of corresponding points of*

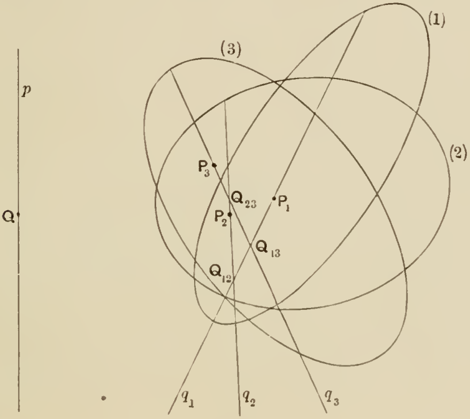
(i) *an involution on a tangent to the conic, is a straight line,*

(ii) *two superposed projective ranges on a tangent to the conic, is a conic having double contact with the given conic,*

(iii) *an involution, is a conic through the double points of the involution,*

(iv) *two superposed projective ranges, is a curve of the fourth degree.*

(2) *The locus of the common conjugates of three conics is a cubic.*



Let $P_1$, $P_2$, $P_3$ be the poles of any line $p$ with regard to the conics (1), (2) and (3) respectively. Let the polars of any point $Q$ on $p$ with respect to (1), (2) and (3) respectively be $q_1$, $q_2$, $q_3$. If $q_1$ and $q_2$ intersect in $Q_{12}$ and $q_1$ and $q_3$ intersect in $Q_{13}$ the loci of $Q_{12}$ and $Q_{13}$, for different positions of $Q$ on $p$, will be the loci of common conjugates of points on $p$ with respect to (1), (2) and (1), (3) respectively. These loci are conics (Art. 117 (2)). They will intersect in four points of which $P_1$ is one. At their points of intersection other than $P_1$ the three polars of points on $p$ with respect to (1), (2) and (3) meet at a point. Hence there are three points on $p$ the polars of which with respect to (1), (2) and (3) meet at a point. Therefore there are three points on $p$ which have common conjugates with regard to the three conics. Hence it may be inferred that the line $p$ meets the locus of common conjugates in three points and that the locus is therefore a cubic.

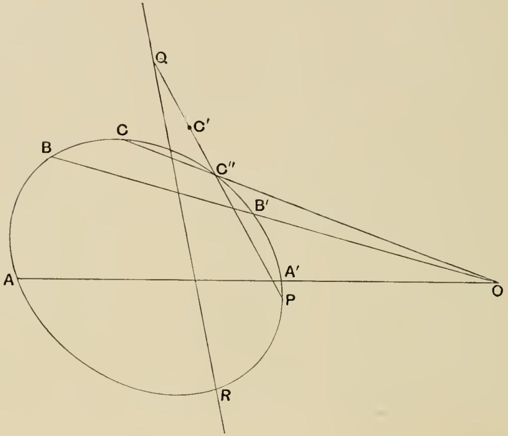
If $C_{12}$, $C_{13}$, $C_{23}$ be the loci of common conjugates of points on $p$ with respect to (1), (2); (1), (3) and (2), (3) respectively, these conics will pass through the three common conjugates of points on $p$, their other points of intersection in pairs being the points $P_1$, $P_2$ and $P_3$.

If the three conics pass through two points $A$ and $B$, the line $AB$ is part of the locus. Hence in this case the cubic becomes a straight line and a conic, which also passes through $A$ and $B$.

Hence *if three conics pass through two points A and B, the locus of their common conjugates is a conic through A and B together with the line AB.*

This result should be compared with those of Arts. 88 (*c*) and 132 (*e*) of which it is the general case.

(3) *To find the nature of the locus of a point such that its connectors to three pairs of coplanar points form an involution pencil.*



Let $A$, $A'$ ; $B$, $B'$ ; $C$, $C'$ be the three pairs of points. The locus passes through each of these points. For through any one of them $C$, a line can be drawn which is a conjugate of $CC'$ in the involution determined by $A$, $A'$ and $B$, $B'$ at $C$. This line is a tangent at $C$ to the locus.

Describe a conic through any five of the six points, viz. $A$, $A'$, $B$, $B'$ and $C$. Let $AA'$ and $BB'$ intersect at $O$. Join $CO$ to meet the conic at $C''$. Let $C'C''$ meet the conic at $P$, and the polar of $O$ with respect to the conic, *i.e.* the connector of $AB' . BA'$ and $AB . A'B'$, at $Q$.

Consider the points of the locus, which are situated on $PQ$. For any point on $PQ$ the pencil determined by $A$, $A'$, $B$, $B'$, $C$, $C'$ will generally be the same as the pencil determined by $A$, $A'$, $B$, $B'$, $C$, $C''$. Hence generally, if one is an involution pencil, so is the other. There are however two exceptions. At $C''$ the involution pencil determined by $AA'$, $BB'$, $CC''$ consists of the connectors of the first five points to $C''$ together with the tangent to the conic at $C''$. Therefore $C''$ is not a point on the locus. The locus passes through $C'$ and the involution pencil determined at $C'$ consists of the connectors of $C'$ to $A$, $A'$, $B$, $B'$, $C$ and the tangent to the locus at $C'$. At all other points on $PQ$ the pencils are the same.

The pencil obtained by joining a variable point to $AA'$, $BB'$, $CC''$ will not be an involution pencil unless the variable point is situated on the conic or on the polar of $O$ with respect to the conic (Art. 98). Hence the only points beside $C'$, in which $PQ$ meets the locus, are $P$ and $Q$. Therefore the locus is met by the line $PQ$ in three and only three points, viz. $C'$, $P$ and $Q$, and is therefore a cubic curve.

The same result may also be obtained by considering the points, in which the locus is met by the connector of any pair of the given pairs of points—say $AA'$. The locus meets this line in $A$ and $A'$. If $T$ be any other point of the locus situated on $AA'$, the line $AA'$ must be a double ray of the involution determined at $T$ by $B$, $B'$ and $C$, $C'$. Let $CC'$ meet $AA'$ in $O'$. Take $O_1$ and $O_1'$ the harmonic conjugates of $O$ and $O'$ with respect to $BB'$ and $CC'$ respectively. Then $O_1O_1'$ meets $AA'$ in $T$. Since this construction for $T$ is unique the locus intersects $AA'$ in only three points, viz. $A$, $A'$ and $T$, and is therefore a cubic curve.

When the six points are on a conic and the connectors of the pairs of points are concurrent at a point $O$, the cubic becomes the conic through the six points and the polar of $O$ with respect to the conic (Art. 98).

If $C'$ is situated on the conic, $C''C''$ is the tangent to the locus at $C'$ and $C'C''$ meets the locus in a third point at $Q$.

The cubic curve obtained as the above locus is the most general form of cubic curve and from a projective point of view a cubic curve may with advantage be defined from this involution property. The student who is desirous of continuing his studies in the projective properties of cubic curves is referred to the work by Schröter, *Theorie der ebenen Kurven dritter ordnung*, on this subject.

If the left-hand side of the equations of the lines $AC'$, $C'B$, $BA'$, $A'C$, $CB'$, $B'A$, in the form $x\cos\alpha + y\sin\alpha - p = 0$, be respectively denoted by $u$, $w'$, $v$, $u'$, $w$, $v'$, and the sides of the hexagon so formed by $a$, $c'$, $b$, $a'$, $c$, $b'$, respectively, then from the equality of the anharmonic ratios of an involution pencil, the equation of the cubic can be obtained in the form

$$a\,.\,b\,.\,c\,.\,u\,.\,v\,.\,w = a'\,.\,b'\,.\,c'\,.\,u'\,.\,v'\,.\,w'.$$

The following examples refer to some of the properties of the cubic considered from this point of view.

## EXAMPLES.

(1) If $AB\,.\,A'B'$ and $AB'\,.\,BA'$ be denoted by $C''$, $C'''$ and points $A''$, $A'''$ and $B''$, $B'''$ be constructed in a similar manner, prove that the cubic passes through $A''$, $A'''$, $B''$, $B'''$, $C''$, $C'''$, and that the pairs of points $A''A'''$, $B''B'''$, $C''C'''$ are related to the cubic in the same way as $AA'$, $BB'$, $CC'$.

(2) Prove that the cubic passes through the points $C''A''\,.\,C'''A'''$, $C''A'''\,.\,C'''A''$, and the four other points found in a similar manner.

(3) If $P$ be a point on the cubic and any pair of conjugate rays in the involution pencil $(P\,.\,AA'BB'CC')$ meet the cubic in $D$ and $D'$, prove that the given cubic meets the cubic determined by $AA'$, $BB'$, $DD'$ in nine determined points.

(4) Prove that the tangents at $C$ and $C'$ are the conjugates of $CC'$ in the involutions determined by $AA'$, $BB'$ at $C$ and $C'$.

(5) Construct the third point in which the tangent at any one of the six points meets the cubic.

### m to n correspondence.

**143**. In the consideration of two ranges of points situated either on given straight lines or on conics it may happen that to a point of the first correspond $n$ points of the second and that to a point of the second correspond $m$ points of the first. This is spoken of as $m$ to $n$ correspondence. It may be defined as follows, viz.

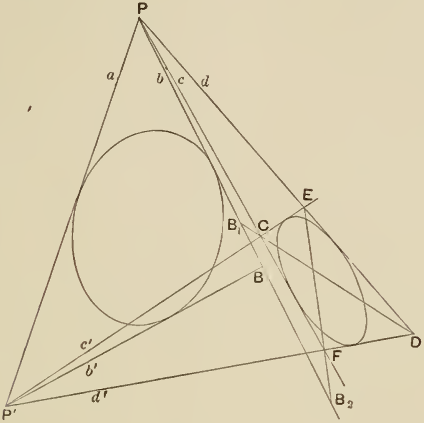
*When two groups of geometrical forms (points, straight lines, curves) are so related that a set of* $m$ *elements of one group corresponds to a set of* $n$ *elements of the other, in such a way that any element of either set determines all the elements of the other set, then the elements of the groups are said to have* $m$ *to* $n$ *correspondence.*

Thus, if two systems or pencils of conics are described through two sets of four points $A$, $B$, $C$, $D$ and $A'$, $B'$, $C'$, $D'$ respectively, a conic of either system is determined, if the tangent at $A$ or $A'$ to the conic is given. If there is one to one correspondence between the tangents at $A$ and $A'$, there is one to one correspondence between the conics. The conics of the two systems determine involutions $P$ and $Q$ on any straight line in that plane. Any point $P$ of one involution determines two points of the involution $Q$ and any point $Q$ determines two points of the involution $P$. Hence there is two to two correspondence between the points $P$ and $Q$.

In the case of two superposed ranges with algebraic one to one correspondence, it was proved (Art. 41) that there are 2, *i.e.* $1+1$, self-corresponding points. Similarly, *in the case of two superposed ranges with* $m$ *to* $n$ *correspondence there are* $m+n$ *self-corresponding points.* For, if $x$ and $x'$ are the distances of a pair of corresponding points from some fixed point on the base, there must be a relation $F(x, x')=0$ between these quantities, where $F(x, x')$ is a polynomial in $x$ and $x'$. Since this relation is algebraic and $n$ points of the second range correspond to a given point of the first, it must be of degree $n$ in $x'$. Similarly, it is of degree $m$ in $x$. To obtain the self-corresponding points of the ranges $x'$ must be made equal to $x$ and the equation thereby obtained is of the $(m+n)$th degree. Hence there are $m+n$ self-corresponding points of the given ranges.

*To find the locus of points the tangents from which to two given conics form a pencil of constant anharmonic ratio.*

Let $a$, $b$ and $c$, $d$ be the tangents from a point $P$ to the two conics and let the anharmonic ratio $(abcd)$ be $\lambda$.



Take any other point $P'$ on $a$. Draw tangents $b'$, $c'$ and $d'$ to the conics, and let $bb'$ be $B$; $cc'$, $C$; $dd'$, $D$; $c'd$, $E$ and $d'c$, $F$.

If the tangents from $P'$ have the same anharmonic ratio as those from $P$, the points $B$, $C$, $D$ or the points $B$, $E$, $F$ must be collinear. Consider the ranges determined on $b$ by $B$ and by $EF$ and $CD$ for different positions of $P'$. The self-corresponding points of these ranges give the positions of $P'$ for which the pencil $ab'c'd'$ has the value $\lambda$. Let the points of intersection of $CD$ and $EF$ with $b$ be $B_1$ and $B_2$. If $B_1$ is given, $P'$ is found by joining $B_1$ to $CD . EF$, the fixed pole of $a$ with respect to the second conic, by a line which meets $c$ and $d$ in $C$ and $D$. $P'$ is then the point of intersection of the second tangent from $C$ or $D$ with $a$. Therefore, disregarding the fixed point $P$, there is a two to one correspondence on $b$. Hence there are three self-corresponding points of the ranges on $b$ and therefore three positions of $P'$ independent of $P$. Hence the line $PP'$ is met in four points including the point $P$ by the locus, which is therefore of the fourth degree.

This result should be compared with that obtained in Art. 133.

### Involution of the nth order.

*When a group of geometric forms may be divided into sets of n forms and any one of each such set determines the remaining* $n-1$ *elements of the set uniquely, then the different sets of forms are said to constitute an involution of the nth order.*

Thus

(1) A ray $AP$, through a fixed point $A$ on a fixed conic, determines a point $P$ on the conic: the connector of $P$ to a fixed point $O$, not on the conic, determines another point $P'$ on the conic. The construction shows that $AP$ and $AP'$ mutually determine each other and therefore form an involution of the second order. The rays $OP$ and $AP$ are in one to two correspondence.

(2) A conic of a pencil of conics through four fixed points $A, B, C, D$ determines two points $P, P'$ on a fixed straight line, and either $P$ or $P'$ determines by means of the conic $P'$ or $P$. The construction shows that $P, P'$ are elements of an involution of the second order. Also a tangent $AT$ to the conic at $A$ is in one to two correspondence with $P$ and in one to one correspondence with the conic.

(3) A cubic of a pencil of cubics through eight fixed points determines three points $P, P', P''$ on a fixed straight line, and any one of the points $P, P', P''$ determines by means of the cubic the other two points. Hence $P, P', P''$ are elements of an involution of the third order.

It follows from the definition of an involution that *if two involutions of the mth and nth order are in one to one correspondence, then there is an m to n correspondence between the elements, which constitute the involutions.*

*An involution of the nth order has* $2(n-1)$ *united corresponding elements.*

Let $x_1, x_2, x_3, \ldots, x_n$ be coordinates, which determine the positions of a set of $n$ elements of the involution. When any one of these is given, the other elements of the set are uniquely determined. Consequently an algebraic relation of the form $F(x_1, x_2) = 0$ exists connecting the coordinates of any pair of elements. If $x_1$ is given the other $n-1$ elements are determined and, since $x_2$ may be the coordinate of any one of these elements, the expression $F(x_1, x_2)$ must be of the $(n-1)$th degree in $x_2$. Similarly it is of the $(n-1)$th degree in $x_1$. To obtain the united corresponding elements of the involution $x_1$ and $x_2$ in the equation $F(x_1, x_2) = 0$ must be made equal to some quantity $x$. The equation thus obtained is of degree $2(n-1)$ in $x$ and therefore there are $2(n-1)$ united corresponding elements of an involution of the $n$th order.

## Characteristics of Systems of Conics.

**144.** In the preceding pages numerous instances have occurred of conics which comply with certain conditions. Such conditions may be classified as one-fold, two-fold, three-fold, four-fold and five-fold.

Thus, if a conic is required to pass through a given point, or touch a given line, or have a given pair of points for conjugate points, it is said to have a one-fold condition imposed on it.

If a conic is required to touch a given line at a given point, or to determine a given involution on a given line, or to have a given point for the pole of a given line, it is said to have a two-fold condition imposed on it.

If a conic is required to be inscribed in, or circumscribed about, a given triangle, or to have a given triangle for a self-conjugate triangle, it is said to have a three-fold condition imposed on it, and so on.

In every case a two-fold, three-fold, four-fold ... condition may be looked upon as a combination of two, three, four, etc. one-fold conditions. This is illustrated by the instances given above. Conversely, a combination of two, three, four, etc. one-fold conditions may be looked upon as a manifold condition.

A conic may be described to satisfy five one-fold conditions or, what is the same thing, to satisfy a five-fold condition. The number of conics which can be described to satisfy this number of independent conditions is a definite and finite number—depending on the nature of the conditions—which can in each case be ascertained.

A system of conics, that is an infinite number of conics, can be described to satisfy a four-fold condition. If an additional condition be imposed, either that the conics should (*a*) pass through a given point, or (*b*) touch a given line, the number of conics complying with the condition is a definite and finite number, which in the first case will be denoted by $\mu$ and in the second by $\nu$. The numbers $\mu$ and $\nu$ are termed the characteristics of the system.

The characteristics of systems of conics through points and touching lines may be obtained from considering the number of conics passing through five points, through four points and touching one line, through three points and touching two lines, ... etc., which will be denoted by (:·:), (::/), (˙.˙//), ....

From Art. 113,

(:·:) = 1,  (::/) = 2,  (˙.˙//) = 4,
(:///) = 4,  (·////) = 2,  (/////) = 1.

By the principle of duality it is clear that

$$(:\cdot:)=(/////),\quad (::/)=(\cdot////),\quad (\because//)=(:///).$$

Hence for the system of conics $(::)$, etc. the following values of $\mu$ and $\nu$ are obtained, viz.

| System of Conics | $\mu$ | $\nu$ |
|---|---|---|
| $(::)$ | 1 | 2 |
| $(\because/)$ | 2 | 4 |
| $(://)$ | 4 | 4 |
| $(\cdot///)$ | 4 | 2 |
| $(////)$ | 2 | 1 |

Let $C$ denote that a pair of conjugate points with respect to the conic are given.

Let $c$ denote that a pair of conjugate lines with respect to the conic are given.

Let $I_2$ denote that an involution determined by the conic on a straight line is given.

Let $i_2$ denote that an involution pencil determined by the conic is given.

Let $/\!\!\bullet_2$ denote that the conic touches a given line at a given point.

Let $p_2$ denote that a given point and line are pole and polar with respect to the conic.

Let $p_3$ denote that a given triangle is self-polar with respect to the conic.

In the above the subscript letters denote the manifoldness of the condition imposed on the conic. The relations $c$, $i_2$ are the correlatives of $C$ and $I_2$, while $/\!\!\bullet_2$, $p_2$ and $p_3$ are their own correlatives.

On reference to Chapter XVII it will be seen that

$$(\therefore\, /\!\!\bullet_2)=1 \qquad (///\!\!\bullet_2)=1,$$

$$(\cdot\, /\!\!\bullet_2 /\!\!\bullet_2)=1 \qquad (//\!\!\bullet_2 /\!\!\bullet_2)=1,$$

$$(:\, /\!\!\bullet_2 /)=2 \qquad (//\!\!\bullet_2\, \cdot)=2.$$

$$(p_2 \therefore)=1 \qquad (p_2///)=1,$$

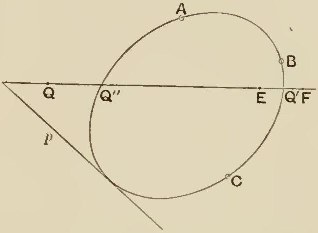
$$(p_2:/)=2 \qquad (p_2\cdot//)=2.$$

$$(p_3:)=1 \qquad (p_3\cdot/)=2 \qquad (p_3//)=1.$$

$$(I_2 \therefore)=1 \qquad (i_2///)=1,$$

and from Chapter XXII $\quad (I_2 I_2\cdot)=1 \qquad (i_2 i_2/)=1.$

It may be proved that in the above $(C)$ may be substituted for $(\cdot)$ and $(c)$ for $(/)$. The proof is general, but for the sake of definiteness it is given for the number of conics, which may be described to pass through three given points, touch a line and have a given pair of points for conjugate points.



Let the points be $A$, $B$, $C$, the line $p$ and the pair of conjugate points $E$ and $F$. Take any point $Q$ on $EF$, and let $Q'$ be the harmonic conjugate of $Q$ with regard to $E$ and $F$. Let $\lambda$ be the number of conics which can be described through $A$, $B$, $C$, $Q'$ to touch $p$. There will then be $\lambda$ points $Q''$ on $EF$ corresponding to $Q'$. Again, to any one of these points $Q''$ there will be $\lambda$ points $Q'$. Hence the correspondence between $Q'$ and $Q''$ and therefore between $Q$ and $Q''$ is $\lambda$ to $\lambda$. Hence there are $2\lambda$ cases in which $Q$ will be the harmonic conjugate of $Q'$ with respect to $E$ and $F$. But the conics will pass through these points in pairs, since $QQ'EF$ is harmonic. Hence $2\lambda$ must be divided by 2 and the number of solutions is $\lambda$.

It was proved in Chapter XVII that

$$(C::)=1 \quad \text{and} \quad \left(c\,{}^{//}_{//}\right)=1,$$

which is consistent with the above result.

By the above any $(\cdot)$ may be replaced by a $(C)$ and afterwards any other $(\cdot)$ may be replaced by another $(C)$, so that $(\cdot)$ and $(C)$ are in all cases interchangeable. This also applies to $(/)$ and $(c)$.

These results likewise follow from Art. 114 $(f)$.

Similarly for $(I_2)$ and $(i_2)$ may be substituted $(:)$ and $(//)$.

### 145. Limiting Forms of Conics.

If a conic degenerate into a pair of straight lines, it is spoken of as a line-pair, and, if into a pair of points, as a point-pair. Under certain circumstances it may become what is termed a line-pair-point. The connection of these with the conic has been stated by Professor Cayley as follows:

> "A Conic is a curve of the second order and second class: quà curve of the second order it may degenerate into a pair of lines or line-pair (but the class is then $=0$); quà curve of the second class it may degenerate into a pair of points, or

point-pair (but the order is then $=0$). The two lines of a line-pair may be coincident, and we have then a coincident line-pair; such a line-pair (it may I think be postulated) ordinarily arises, not from a line-pair the two lines of which become coincident, but from a proper conic, flattening by the gradual diminution of its conjugate axis, while its transverse axis remains constant or approaches a limit different from zero; the conic thus tends (not to an indefinitely extended but) to a terminated line*; in other words, the tangents to the Conics become more and more nearly lines through two fixed points, the terminations of the terminated line; and these terminating points, which continue to exist up to the instant when the conjugate axis takes its limiting value $=0$, are regarded as still existing at this instant, and the coincident line-pair as being in fact the point-pair formed by the two terminating points. Similarly, the two points of a point-pair may be coincident, and we have then a coincident point-pair; such a point-pair (it must in like manner be postulated) ordinarily arises, not from a point-pair the two points of which become coincident, but from a proper conic sharpening itself to coincide with its asymptotes and so becoming ultimately a pair of lines through the coincident point-pair; and the coincident point-pair is regarded as being in fact the line-pair formed by the same two lines through the coincident point-pair.

"In accordance with the foregoing notions we may with propriety, and it will in the sequel be found convenient to speak of a point-pair as a line terminated by two points in this line, and similarly to speak of a line-pair as a point terminated (that is the pencil of lines through the point is terminated) by two lines through the point.

"If in a point-pair thus considered as a line terminated by two points the two points become coincident (the line continuing to exist as a definite line), or, what is the same thing, if in a line-pair thus considered as a point terminated by two lines the two lines become coincident (the point continuing to exist as a definite point), we have a line-pair-point; viz. this is at once a coincident line-pair and a coincident point-pair; it may also be regarded as the limit of a conic the axes of which, and the ratio of the conjugate to the transverse axis, all ultimately vanish: it may be described as a line terminated each way at a point thereof, or as a point terminated each way at a line through it.

"* A line is regarded as extending from any point $A$ thereof to $B$, and then in the same direction, from $B$ through infinity to $A$; it thus consists of two portions separated by these points; and considering either portion removed, the remaining portion is a terminated line."

A conic which is a line-pair considered as an envelope consists of two points united at the point of intersection of the line-pair, and a conic which is a point-pair regarded as a locus consists of two straight lines united in the connector of the point-pair.

### 146. Properties of a System of Conics of characteristics $\mu$ and $\nu$.

(1) The locus of the pole of a given line is a curve of degree $\nu$.

Consider in how many points the locus meets the given line. Where it meets it the pole of the given line is on the given line, which must therefore be a tangent at that point to some conic of the system. But there are $\nu$ such points. Hence the locus meets the given line in $\nu$ points and is of degree $\nu$.

(2) The locus of the points of intersection of tangents from two fixed points to conics of the system is of degree $3\nu$.

Let $K$, $L$ be the two fixed points. The number of points in which this line is met by the locus will determine the degree of the locus. For a point $P$ on $KL$ to be on the locus a conic of the system must touch $KL$ at $P$. Then one pair of tangents intersect at $P$, another at $K$, and a third at $L$. Hence for every conic which touches $KL$ there are three points on $KL$. Therefore the locus is of degree $3\nu$ and $K$ and $L$ are each double points on the locus of order $\nu$.

(3) The envelope of the polars of a given point is a curve of order $\mu$.

This is the correlative of (1).

(4) The locus of a point such that its polar with respect to a fixed conic coincides with its polar with respect to a conic of the system is of degree $\mu+\nu$.

Consider in how many points the locus meets a given line $l$. Take two points $P$ and $P'$ on $l$ such that $p$ is the polar of $P$ with regard to the fixed conic and the polar of $P'$ with regard to a conic of the system.

If $P$ be fixed, $p$ is fixed and by (1) the locus of its poles with respect to the system of conics is a curve of degree $\nu$ which determines $\nu$ positions of $P'$ on $l$.

If $P'$ is fixed, by (3) its polars with respect to the system of conics envelope a curve of class $\mu$. From $Q$ the pole of $l$ with respect to the fixed conic $\mu$ tangents can therefore be drawn to this locus. Therefore there are $\mu$ polars of $P'$, which pass through $Q$, and their poles with respect to the fixed conic are on $l$. Hence there are $\mu$ positions of $P$ on $l$.

Therefore the correspondence between $P$ and $P'$ is a $\mu$ to $\nu$ correspondence, and the locus meets $l$ in $\mu+\nu$ points and is of degree $\mu+\nu$.

(5) The number of conics which touch a fixed conic is $2(\mu+\nu)$.

If a conic of the system touch the fixed conic, the point of contact has the same polar with respect to both conics and is therefore on the locus in (4). The points of contact are therefore the intersections of the fixed conic with the locus in (4). There are therefore $2(\mu+\nu)$ points of contact and the same number of conics touching the given one.

(6) Verify that there are $2\nu-\mu$ line-pairs and $2\mu-\nu$ point-pairs in the system.

## EXAMPLES.

(1) The locus of the foci of a system of conics touching four straight lines is a cubic passing through the circular points at infinity.

This follows from (2).

(2) If in (2) one of the conditions is that the conic should touch $KL$, the degree of the locus is $2\nu$, $K$ and $L$ being double points of order $\nu$.

(3) A conic section touches three given straight lines and has one of its principal axes in a given direction; show that the locus of its foci is a cubic curve which passes through the angular points of the triangle formed by the given lines.

### Pascal Hexagons for six points on a conic.

**147.** In Art. 100 it was proved that, if any six points be taken on a conic, the three points of intersection of the pairs of opposite sides of the hexagon so formed lie on a straight line termed the Pascal line of the hexagon in question.

*Notation.* The vertices of the Pascal hexagon will be denoted by the numbers 1, 2, 3, 4, 5, 6. The Pascal line of this hexagon is the connector of 12.45 ; 34.61 ; and 56.23. If the points are written in reverse order the hexagon is unaltered, thus (346521) is the same as (125643). Also any number of points may be transferred from the beginning to the end of the cycle without altering the hexagon. Thus (564123) is the same as (412356).

If three lines are selected by taking the numbers in any pairs, say 35, 41, 26, and a second selection is made in a similar way, say 21, 36, 54, and this system of lines is combined with the former, thus

$$\left\{\begin{matrix}35\\21\end{matrix}\right. \quad \left\{\begin{matrix}41\\36\end{matrix}\right. \quad \left\{\begin{matrix}26\\54\end{matrix}\right.$$

in such a way that no combination contains the same number twice, then a hexagon may always be formed of which the points of intersection of the pairs of lines are the Pascal line.

Thus in the instance in question, consider $\left\{\begin{matrix}35\\21\end{matrix}\right.$ in the first place. Write down 3, 5, 2, 1 as the 1st, 2nd, 4th, and 5th elements—thus (35.21.). In the next combination there is 41. Place a 4 after the 1—thus (35.214). Corresponding to 41 there occurs the line 36. Interchange the 3 and 5—thus (53.214); then insert 6 as the third element. In this way (536214) is obtained. Then 26 and 54 are the remaining pair of opposite sides.

(1) *With the same* 6 *points, which may be called the P points, as vertices* 60 *Pascal hexagons can be formed, and corresponding to these hexagons there are* 60 *different Pascal lines, which may be termed the p lines.*

Taking one of the points as the first vertex, the second may be selected in 5 ways, the third in 4, the fourth in 3, the fifth in 2 and the sixth in 1 way. Thus 120 hexagons are obtained, but as each occurs with its vertices in reverse order there are 60 hexagons.

(2) *There are* 15 *connectors of the* 6 *points, termed c lines, which intersect in* 45 *points, termed T points, in addition to the* 6 *P points.*

A connector $c$ is obtained by joining any one of the 6 $P$ points to any of the remaining 5 $P$ points. In this way 30 $c$ connectors are obtained, but each of these connectors is counted twice, *e.g.* 12 and 21 are the same line. Hence the number of connectors is 15. Each of the connectors $c$ meets 4 of the other connectors at each of the $P$ points situated on it. These account for 8 of its 14 intersections with $c$ lines. There are therefore 6 other $T$ points on each connector. Hence, as each $T$ point is counted twice, there are 45 $T$ points.

(3) *Through each of the* 45 *T points* 4 *Pascal lines pass, and each Pascal line passes through* 3 *T points.*

Let the points $P$ be denoted by the numbers 1, 2, 3, 4, 5, 6. Then the Pascal lines of the hexagons (123456), (123546), (126453) and (126543) all pass through the point 12.45.

These hexagons are obtained by keeping one consecutive pair of elements 12 unaltered and in turn changing among themselves the elements of the opposite pair and those of the third or remaining pair.

By Pascal's theorem each Pascal line passes through 3 $T$ points.

From this the number of Pascal lines can be independently obtained for $45 \times 4 = 180$. But by Pascal's theorem, a Pascal line goes through 3 $T$ points and the number of Pascal lines is therefore $180 \div 3 = 60$.

(4) *If three alternate vertices of a Pascal hexagon are the same and the other vertices are deduced by interchanging the other three vertices in cyclic order, the three Pascal lines of the hexagons so obtained are concurrent in a point termed a G point.*

Consider (123456) (163254) (143652).

Their Pascal lines are

$$\begin{array}{ccc} \underline{12 \,.\, 45} & \underline{16 \,.\, 25} & 14 \,.\, 65 \\ 23 \,.\, 56 & \underline{63 \,.\, 54} & \underline{43 \,.\, 52} \\ \underline{34 \,.\, 61} & 32 \,.\, 41 & \underline{36 \,.\, 21} \end{array}$$

Consider the points of these which are underlined.

Three of these are the vertices of a triangle 12 . 63 . 45.

The other three „ „ „ „ 43 . 52 . 61.

The three points of intersection of pairs of corresponding sides of these triangles are

$$\left\{\begin{matrix}12\\43\end{matrix}\right. \qquad \left\{\begin{matrix}63\\52\end{matrix}\right. \qquad \left\{\begin{matrix}45\\61\end{matrix}\right.$$

Since they are on the Pascal line of (125436) they are collinear. Hence the connectors of the vertices of the triangles are concurrent, that is the three given Pascal lines are concurrent.

If the first hexagon had been (163452), the other hexagons would have been (123654) and (143256), and their Pascal lines would have given another $G$ point. Hence *corresponding to any three fixed vertices there are two G points, which may be termed conjugate G points.*

(5) *There are* 20 *G points at each of which* 3 *Pascal lines intersect.*

Taking any vertex for the first of the fixed vertices, the second of the fixed vertices may be selected in 5 ways and the third in 4 ways. This gives 20 arrangements of the fixed vertices. But if the arrangements are read in opposite order the hexagons are not altered. Hence this number must be divided by 2. For each order there are 2 $G$ points. Hence the total number of $G$ points is 20.

(6) *If on a Pascal line any pair of opposite elements are taken, and these are interchanged in turn with the pairs of adjacent elements, so that they themselves become adjacent elements, then the two Pascal lines so obtained are concurrent with the given Pascal line in an H point; and as this transposition may be accomplished in* 3 *ways, there are* 3 *H points on every Pascal line.*

Consider $ABC$ the Pascal line of (123456).

Take 12, 34, 56 three of the sides of this hexagon, each of which passes through a different point $A$, $B$, $C$, and which form a triangle (1).

Through each of $A$, $B$ and $C$ three other Pascal lines pass (3). If any one line be taken out of each of these groups, a triangle is formed which is in perspective with the triangle 12, 34, 56, with $ABC$ for axis of perspective.

Three such Pascal lines are

$$P(213546) \quad P(624351) \quad P(132465).$$

The first is obtained from (123456) by interchanging the 12 and the pair of opposite vertices. The second and third are obtained in a similar manner by interchanging 34 and 56 and the pairs of opposite vertices. The triangle formed by these lines may be termed the triangle (2).

Since the triangles (1) and (2) are in perspective, the lines joining their corresponding vertices are concurrent.

But the point of intersection of

$P$ (624351) and $P$ (132465) is 24 . 51
$P$ (132465) and $P$ (213546) is 46 . 13
$P$ (213546) and $P$ (624351) is 62 . 35.

Therefore the connectors of 34 . 56 to 24 . 51
56 . 12 to 46 . 13
12 . 34 to 62 . 35

are concurrent.

But these lines are the Pascals of

(342651), (564213), (126435).

The second and third Pascal lines are obtained from the first by interchanging according to the rule the pair of opposite elements 2 and 1. From (5) and (6) it follows that *on every Pascal line there are* 3 *H points and one G point.*

(7) *There are* 60 *H points, through each of which three Pascal lines pass, and three H points are situated on each Pascal line.*

This follows from (6).

(8) *The* 60 *p lines may be divided into* 6 *systems of* 10 *p lines, such that the p lines of each system intersect each other in threes in the three H points which are situated on these lines so determining* 10 *H points.*

Consider any three Pascal lines $P$(123456), $P$(214356), $P$(623541), which intersect in an $H$ point. The other $H$ points on these Pascal lines are the points of intersection of these Pascals with six other Pascals, viz.

$P$(213465), $P$(132546), $P$(625314),
$P$(124365), $P$(632451), $P$(263514).

These form two triangles in perspective and the points of intersection of corresponding sides are three $H$ points, which lie on the Pascal line $P$(425136). These ten Pascal lines form the system.

On each of the 10 $p$ lines 3 $H$ points have been formed and therefore there can be no further $H$ points on these lines. None of the other points of intersection of these 10 $p$ lines give $G$ points or $T$ points. Hence the three $p$ lines which meet at a $G$ point and the four $p$ lines which meet at a $T$ point all belong to different systems.

The Pascal lines of any system may be obtained by writing down any Pascal and interchanging any number of times the elements in any two adjacent pairs. The six Pascals through a $G$ point and the conjugate $G$ point thus determine the six different systems.

(9) *There are* 20 $g$ *lines each of which passes through one* $G$ *point and three* $H$ *points.*

Let the triangle formed by the three Pascal lines

$$P(124635) \quad P(513462) \quad P(241356)$$

and termed triangle (3) be associated with the triangles (1) and (2) of (6).

Each Pascal line is obtained from the corresponding one in (2), by interchanging the elements of the pairs 12, 34, 56, and afterwards interchanging the other two pairs of elements.

These lines intersect in the points

$$13 . 62$$

$$35 . 24$$

and

$$46 . 51.$$

Their connectors respectively to

$$34 . 56$$

$$56 . 12$$

$$12 . 34$$

are the Pascal lines of

$$(342651), \quad (564213), \quad (126435).$$

Hence the three triangles (1), (2) and (3) have a common centre of perspective, viz. the $H$ point of (6).

Hence the corresponding sides of (2) and (3) intersect on a straight line, *i.e.* the points

$$P(213546) . P(124635)$$

$$P(624351) . P(513462)$$

$$P(132465) . P(241356)$$

are collinear. But these are 3 $H$ points which are therefore collinear.

The axis of perspective of the triangles (2) and (3) passes through the point of intersection of the axes of perspective of (1) and (2) and of (1) and (3). (Art. 33 (ii).) The axis of perspective of (1) and (2) is $P(123456)$. The axis of perspective of (1) and (3) is the connector of $12 . 63$ ; $34 . 52$ ; $56 . 14$, *i.e.* $P(143652)$. These intersect in a $G$ point which is therefore collinear with the three $H$ points already found.

As there are twenty $G$ points through each of which a $g$ line passes, there are twenty $g$ lines. As there are 60 $H$ points and each $g$ line passes through 3 $H$ points, only one $g$ line passes through each $H$ point.

The $g$ line through a point $G$ on a Pascal line passes through $H$ points on 3 Pascals obtained by interchanging pairs of opposite vertices. The $H$ points on these Pascals are obtained by interchanging the already interchanged vertices with the adjacent elements of the other pairs, so that they become adjacent elements.

Thus the $g$ line through the $G$ point on (123456) passes through the following $H$ points, viz.

$$\left\{\begin{array}{l}423156\\241356\\623514\end{array}\right. \qquad \left\{\begin{array}{l}153426\\135246\\513462\end{array}\right. \qquad \left\{\begin{array}{l}126453\\124635\\362451\end{array}\right.$$

(10) *If a triangle be formed by the connectors of three pairs of the $P$ points, the four Pascal lines through each vertex of this triangle intersect in threes in 4 $G$ points and in threes in 4 $H$ points.*

Consider the triangle formed by 14, 23 and 56 and let 14 . 23, 23 . 56 and 56 . 14 be $C$, $B$ and $A$.

By (3) the Pascals through $A$, $B$ and $C$ are respectively

| | $A$ | $B$ | $C$ |
|---|---|---|---|
| (1) | $P$ (563412) | $P$ (234561) | $P$ (145236) |
| (2) | $P$ (562143) | $P$ (234651) | $P$ (146235) |
| (3) | $P$ (563142) | $P$ (231564) | $P$ (146325) |
| (4) | $P$ (562413) | $P$ (231654) | $P$ (145326) |

The Pascals given in the horizontal columns intersect by (4) in 4 $G$ points.

The Pascal lines may be grouped as follows:

| | $A$ | $B$ | $C$ |
|---|---|---|---|
| (1) | $P$ (563412) | $P$ (234651) | $P$ (146325) |
| (2) | $P$ (562143) | $P$ (234561) | $P$ (145326) |
| (3) | $P$ (563142) | $P$ (231654) | $P$ (145236) |
| (4) | $P$ (562413) | $P$ (231564) | $P$ (146235) |

In this case the Pascals in each row intersect in an $H$ point (6). If 56, 32, 41 are regarded as the pairs of elements and, when they stand consecutively, they are called adjacent, and when not so standing, opposite elements, the following rule is obtained to write down one of the above $H$ points from another.

*In one Pascal, change the adjacent elements, in another the opposite elements, and in the third both the adjacent and the opposite elements.*

The same rule holds for obtaining the above $G$ points one from another.

Any one of the above 12 Pascals meets others of the 12 Pascals in 3 points at $A$, $B$ and $C$; at 2 points at $G$ points, and at 2 points in $H$ points.

There are therefore 4 of these Pascals which meet a given Pascal in other points—these Pascals pass two by two through the two vertices of the triangle $ABC$ through which the given Pascal does not pass.

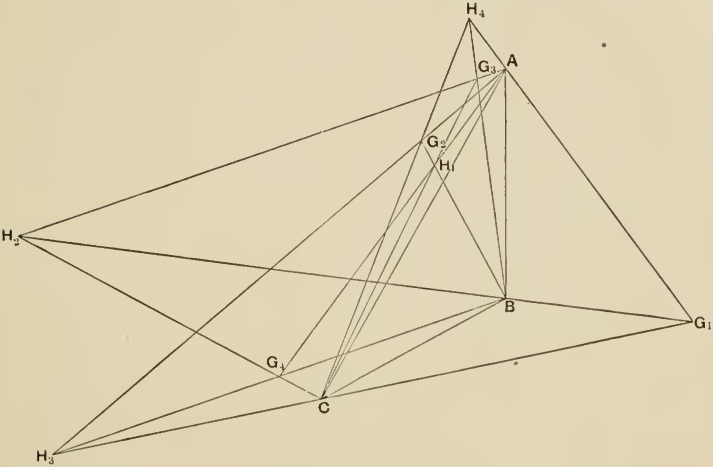
(11) *The four $g$ lines of four $G$ points on Pascals obtained by keeping one pair of adjacent elements fixed and interchanging in turn the elements of the opposite pair and of the third pair are concurrent at an $I$ point.*

Thus it has to be proved that the $g$ lines of the $G$ points on $P$ (563412), $P$ (562143), $P$ (563142) and $P$ (562413) are concurrent. These are the $G$ points of (10).

By (9) these $g$ lines are determined by joining the respective $G$ points to corresponding $H$ points. Thus to obtain these lines it is necessary to join the $G$ point of

$P$ (563412) to $P$ (562413) . $P$ (564231)
$P$ (562143) $P$ (563142) . $P$ (561324)
$P$ (563142) $P$ (562143) . $P$ (561234)
$P$ (562413) $P$ (563412) . $P$ (564321)

But these $H$ points are respectively the $H$ points (4), (3), (2) and (1) of (10).



Hence the $G$ points and $H$ points form two quadrangles which are in perspective in three ways with the vertices of the triangle $ABC$ and the opposite sides as centres and axes of perspective. Hence (Example 4, Art. 33) the lines $G_1H_1$, $G_2H_2$, $G_3H_3$, $G_4H_4$ meet at a point. This is the required $I$ point.

Fifteen such triangles as $ABC$ can be constructed and therefore the twenty $g$ lines pass 4 by 4 through fifteen $I$ points.

(12) *The $G$ points lie 4 by 4 on 15 lines $i$.*

Consider the 3 lines ($a$)

$P$ (126453) $P$ (451362) $P$ (635124)

which meet at an $H$ point.

Take three $T$ points on these lines ($a$), viz.

(46 . 13) (26 . 15) (24 . 35)

which form a triangle (i) whose sides are

$P$ (263514) $P$ (246531) $P$ (315462)

Take the three Pascal lines

$P$ (361524) $P$ (461253) $P$ (134526)

They meet on the lines ($a$) and form a triangle (ii) the vertices of which are $H$ points.

The corresponding sides of the two triangles (i) and (ii) intersect in three collinear points

$$\left\{\begin{matrix} P\ (263514) \\ P\ (361524) \end{matrix}\right. \qquad \left\{\begin{matrix} P\ (246531) \\ P\ (643521) \end{matrix}\right. \qquad \left\{\begin{matrix} P\ (315462) \\ P\ (345261) \end{matrix}\right.$$

which are three $G$ points.

The determining Pascals of these $G$ points may be written

$$\begin{matrix} P\ (615243) & \quad P\ (612534) & \quad P\ (613452) \\ P\ (635142) & \quad P\ (642135) & \quad P\ (623154) \end{matrix}$$

The adjacent elements of the Pascals in the upper row are the combinations of 61, 52, 43. One other such combination may be formed, viz. $P$ (614325) which gives a $G$ point where it is met by $P$ (654123). By symmetry this $G$ point must be on the line joining the other three $G$ points whose determining Pascals are given above. Three groups such as 61, 52, 43 can be selected from the given 6 $P$ points in 15 ways and therefore there are 15 $i$ lines.

If $\equiv$ denotes the word "contains" it will be seen that there are

60 $p$ lines, 20 $g$ lines, 15 $i$ lines,
60 $H$ points, 20 $G$ points, 15 $I$ points,

and that

$$H \equiv 3 \,.\, p, \quad G \equiv 1 \,.\, g, \quad H \equiv 1 \,.\, g, \quad G \equiv 1 \,.\, g + 3 \,.\, p, \quad I \equiv 4 \,.\, g,$$
$$p \equiv 3 \,.\, H, \quad g \equiv 1 \,.\, G, \quad p \equiv 1 \,.\, G, \quad g \equiv 1 \,.\, G + 3 \,.\, H, \quad i \equiv 4 \,.\, G.$$

## EXAMPLES.

(1) In (10) three of the anharmonic ratios of the three pencils of Pascal lines through $A$, $B$, $C$ are such that their product is unity.

(2) Conjugate $G$ points are conjugate points with respect to the conic.

# CHAPTER XXII

## INVOLUTION IN CONNEXION WITH THE CONIC

### 148. Involutions Determined by Conics.

It has been shown Art. 95 ($e$) that given a conic every point in its plane has associated with it a definite line called the polar of the point. If any point $P$ be taken on a line ($l$) its polar will meet the line ($l$) in a point $P'$ and for different positions of $P$, the points $P$ and $P'$ are pairs of points of an involution (Art. 95 ($k$)). For different conics the involutions determined on $l$ will generally be different. If the line $l$ meets the conic in real points, these points are the double points of the involution. If there are no real double points of the involution, the line does not meet the conic in real points. The involution, however, exists in the case of every line and every conic. Theorems, therefore, which deal with the points of intersection of lines and conics may in some cases be extended to theorems dealing with conics and the involutions which they determine on lines.

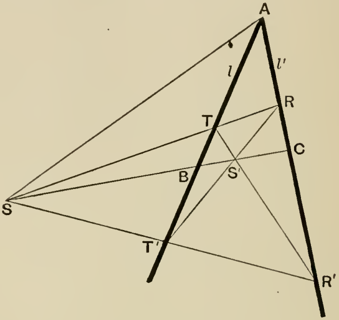
If a line passes through two real points of intersection of two conics, both conics determine the same involution on the line, viz., an involution made up of pairs of points which are harmonic conjugates of the two points in which the line meets the two conics. Hence, if lines are discovered which do not meet conics in real points but on which the conics determine the same involution, there is a strong analogy between such lines and lines joining the points of intersection of the conics.

There are at least two straight lines—termed common involution chords—on which a pair of conics determine the same involution. (Art. 125, Chapter XX.)

In this chapter certain theorems concerning conics and the involutions which they determine on straight lines will be proved.

*If a conic determines two involutions of the same kind on two straight lines $l$ and $l'$ which intersect at $A$ and are in perspective at $S$ and $S'$, then $ASS'$ is a self-conjugate triangle of the conic.*

(1) If the involutions have real double points, the conic passes through these points and $ASS'$ is the diagonal points triangle of an inscribed quadrangle and is therefore a self-conjugate triangle.



(2) If the involutions have imaginary double points, let $B$ and $C$ be the conjugates of $A$ in the involutions; then $BC$ is the polar of $A$, and $S$ and $S'$ are situated on $BC$. Since the involution on $AB$ has imaginary double points, a pair of conjugate points $T$ and $T'$ which are harmonic conjugates of $A$ and $B$ can be found, for they are the common conjugates of two involutions, one with real and one with imaginary double points. (Art. 55.)

Join $T$ and $T'$ to $S'$ to meet $l'$ in $R'$ and $R$.

Join $T'$ to $R'$ to meet $BC$ in $S''$. Then, because $ABTT'$ is harmonic, $CBS'S''$ is harmonic; therefore $S''$ coincides with $S$ the second centre of perspective and $R'T'$ passes through $S$.

Similarly $RT$ passes through $S$.

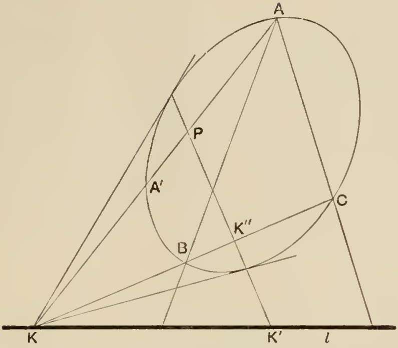
Consider the quadrangle $STS'T'$. The ends of the diagonals $TT'$ and $RR'$ are conjugate points with regard to the conic. Therefore $S$ and $S'$ are also conjugate points, and, since $SS'$ is the polar of $A$, the triangle $ASS'$ is self-conjugate.

$R$ and $R'$ are harmonic conjugates of $A$ and $C$, and $S$ and $S'$ may be found at once from $T$, $T'$ and $R$, $R'$.

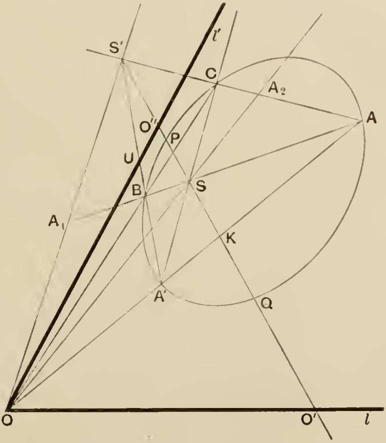
### 149. Methods for describing conics to determine given involutions on given lines.

(1) *To describe a conic through three given points to determine a given involution on a given line.*

Let $A$, $B$, $C$ be the given points and $l$ the given line. Let $BC$ meet $l$ in $K$. Take $K''$ the harmonic conjugate of $K$ with respect to $BC$, and $K'$ the conjugate of $K$ in the given involution. Then $K'K''$ is the polar of $K$. Let $KA$ meet $K'K''$ in $P$. Take $A'$ the harmonic conjugate of $A$ with respect to $KP$. Then $A'$ is a point on the conic. Similarly points $B'$ and $C'$ may be constructed. The required conic is the conic through $A$, $B$, $C$, $A'$, $B'$, $C'$.



(2) *To describe a conic through a given point to determine given involutions on two given lines.*



Let $A$ be the given point and $l$ and $l'$ the lines on which the involutions are situated. Let $l$ and $l'$ meet at $O$ and let $O'$ and $O''$ be the conjugates of $O$ in the two given involutions.

Then $O'O''$ is the polar of $O$. Join $OA$ to meet $O'O''$ in $K$. Take $A'$ the harmonic conjugate of $A$ with respect to $OK$.

Then $A'$ is a point on the curve.

Let $S$ and $S'$ be the two points on $O'O''$ which are centres of perspective for the two involutions (Art. 60). Then the triangle $OSS'$ is self-conjugate with regard to the conic (Art. 148).

Join $A$ to $S$ and $S'$ to meet the opposite sides of this triangle in $A_1$ and $A_2$.

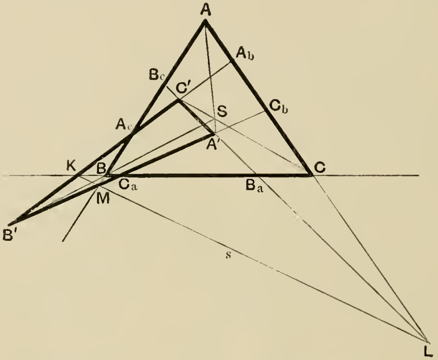
Let $B$ and $C$ be the harmonic conjugates of $A$ with respect to $A_1S$ and $A_2S'$.

Then $B$ and $C$ are on the conic.

Let $BA'$ meet $l'$ in $U$. The polar of $U$ is the line joining its conjugate on $l'$ to its harmonic conjugate with regard to $BA'$. Let this line meet $UA$ in $R$. Let $U'$ be the harmonic conjugate of $A$ with respect to $UR$. Then $U'$ is also on the curve.

Hence the five points $A$, $A'$, $B$, $C$ and $U'$ determine the required conic.

(3) *To find the necessary and sufficient condition that it may be possible to describe a conic* to determine three given involutions on three given straight lines.*



Let $A$, $B$, $C$ be the points of intersection of the given lines.

Let the polar of $A$ be $A_cA_b$.

„ „ $B$ „ $B_cB_a$.

„ „ $C$ „ $C_aC_b$.

These points determine the involutions on the sides of the triangle $ABC$.

If the polars meet at $A'$, $B'$, $C'$, then the triangles $ABC$ and $A'B'C'$ being polar triangles are in perspective. In the figure $S$ is the centre and $KLM$ the axis of perspective. Hence by the converse of Pascal's Theorem a conic can be described through the points of intersections of the non-corresponding sides. Therefore, $A_b$, $A_c$, $B_a$, $B_c$, $C_a$, $C_b$ are on a conic.

This condition may be shown to be sufficient as follows. Let $E_a$, $F_a$; $E_b$, $F_b$; $E_c$, $F_c$ be the pairs of common harmonic conjugates of $BB_a$, $CC_a$; $CC_b$, $AA_b$ and $AA_c$, $BB_c$ respectively. Then (Art. 59) they are pairs of conjugate points of the involutions determined by $BC$, $B_aC_a$; $CA$, $C_bA_b$; $AB$, $A_cB_c$ respectively. Therefore $\frac{BA_c}{AA_c}\cdot\frac{BB_c}{AB_c}=\frac{BE_c}{AE_c}\cdot\frac{BF_c}{AF_c}$ (Art. 53) and two similar relations hold for $E_a$, $F_a$

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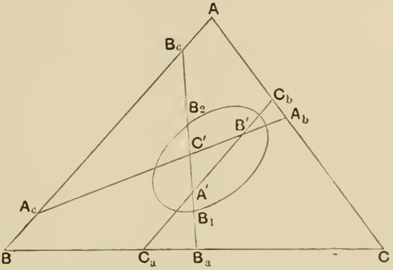
* This conic may be imaginary.

and $E_b$, $F_b$. But Carnot's theorem holds for the points $A_c$, $B_c$, $C_a$, $B_a$, $C_b$, $A_b$ and therefore by substituting from these relations it follows that it holds for the points $E_a$, $F_a$, $E_b$, $F_b$, $E_c$, $F_c$. Hence these six points lie on a conic real or imaginary. The six points $A_c$, $B_c$, $C_a$, $B_a$, $C_b$, $A_b$ can always be associated with the vertices of the triangle in such a way that the conic is real.

(4) *To describe the conic, if real, to determine on given lines three given involutions which satisfy the condition found in* (3).

If any of the involutions have real double points, construct these points and apply the method given in (2).

If none of the involutions have real double points, let $A_cA_b$, $B_aB_c$, $C_aC_b$ be the polars of $A$, $B$, $C$ and let them intersect in $A'$, $B'$, $C'$.



Then $A'$ is the pole of $BC$; therefore $A'$ and $B_a$ are conjugate points. Similarly $C'$ and $B_c$ are conjugate points.

Therefore $B_1$ and $B_2$, the points in which $B_cB_a$ meets the conic, are constructed as the common harmonic conjugates of $A'B_a$ and $C'B_c$ or as the double points of the involution of which these points are pairs of conjugates.

Similarly, the points where $A_cA_b$ and $C_aC_b$ meet the conic are determined. If none of the sides $AB$, $BC$, $CA$ meet the conic in real points and the conic is real, the tangents from $A$, $B$, $C$ must be real and therefore the double points of the three involutions are real. Hence six points on the conic are given.

*If two triangles are in perspective, to construct the conic, if real, with respect to which they are pole and polar triangles.*

The condition set forth in (3) is satisfied, as the triangles are in perspective. Therefore the conic described by (4) to determine on the sides of one triangle the involutions determined on them by the sides of the other triangle is the required conic.

### 150. Anharmonic Property of a conic and the involution which it determines on a given line.

*If a conic determine an involution on a straight line and $A_1$, $B_1$ be the projections on the same straight line of any two fixed points on the conic from a variable point on the conic, then*

$$\frac{\{(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)\}^2}{(X_1X_2Y_1Y_2)}$$

*is constant, where* $X_1X_2$, $Y_1Y_2$ *may be any two pairs of conjugate points of the involution, or if* $X_1X_2$, $Y_1Y_2$ *are two fixed pairs of conjugate points of the involution,* $(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)$ *is constant.*

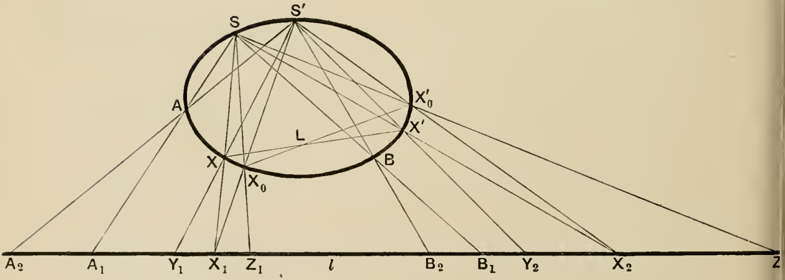
The proof is condensed by determining the positions of the points by means of ratios. If $X_1$, $X_2$ are taken as the points of reference, *i.e.* if their ratio coordinates are 0 and ∞ respectively, $Y_1$, $Y_2$, $A_1$, $B_1$ may be denoted by $y_1$, $y_2$, $a_1$, $b_1$. Then

$$\frac{\{(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)\}^2}{(X_1X_2\,Y_1Y_2)}$$

$$=\frac{\left\{\dfrac{y_2-a_1}{y_2}:\dfrac{b_1-a_1}{b_1}-\dfrac{\infty-a_1}{\infty-y_1}:\dfrac{b_1-a_1}{b_1-y_1}\right\}^2}{\dfrac{y_1}{y_1-\infty}:\dfrac{y_2}{y_2-\infty}}$$

$$=\frac{\left\{\dfrac{b_1}{y_2}\dfrac{y_2-a_1}{b_1-a_1}-\dfrac{b_1-y_1}{b_1-a_1}\right\}^2}{\dfrac{y_1}{y_2}}$$

$$=\left\{\frac{y_1y_2-a_1b_1}{(a_1-b_1)\sqrt{y_1y_2}}\right\}^2.$$



Let $l$ be a line which does not meet the conic in real points and let $L$ be its pole. Take $A$ and $B$ any two fixed points on the conic and $X$ and $X'$ two points on the conic collinear with $L$. The projections of any such pairs of points from any point on the conic upon $l$ give a pair of conjugate points of the involution determined by the conic on $l$.

Project $A, B, X, X'$ from any point $S$ on the conic into $A_1, B_1, X_1, X_2$. Take any point $S'$ on the curve. Join $S'$ to $X_1$, $X_2$ to meet the conic in

$X_0$, $X_0'$. Then, because $X_1$, $X_2$ are conjugate points, $X_0$, $X_0'$ are collinear with $L$ (Art. 95 (*j*)).

Project $X_0$, $X_0'$ from $S$ into $Z_1$, $Z_2$.

Project $A$, $B$, $X$, $X'$ from $S'$ into $A_2$, $B_2$, $Y_1$, $Y_2$.

Then $Z_1Z_2$, $X_1X_2$, $Y_1Y_2$ are pairs of conjugate points of the involution determined on $l$ by the conic and $A_1$, $B_1$, $X_1$, $X_2$, $Z_1$, $Z_2$ are projective with $A_2$, $B_2$, $Y_1$, $Y_2$, $X_1$, $X_2$.

Take $X_1(0)$ and $X_2(\infty)$ as origin and let small letters denote the ratios of the points denoted by the corresponding large letters, then

$$y_1y_2 = z_1z_2 = \text{constant for the involution} \quad \ldots\ldots\ldots\ldots(1),$$

also
$$(X_1X_2Z_1Z_2) = (Y_1Y_2X_1X_2);$$

$$\therefore \frac{z_1}{z_2} = \frac{y_1}{y_2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2);$$

$\therefore$ from (1) $z_1 = -y_1$, $z_2 = -y_2$.

Again
$$(A_1B_1X_1X_2) = (A_2B_2Y_1Y_2);$$

$$\therefore \frac{a_1}{b_1} = \frac{y_1 - a_2}{y_1 - b_2} : \frac{y_2 - a_2}{y_2 - b_2} = \frac{y_1y_2 + a_2b_2 - y_1b_2 + a_2y_2}{y_1y_2 + a_2b_2 - y_1a_2 + b_2y_2};$$

$$\begin{aligned}\therefore (a_1 - b_1)(y_1y_2 + a_2b_2) &= a_1(y_1a_2 + b_2y_2) - b_1(y_1b_2 + a_2y_2)\\ &= y_1(a_1a_2 - b_1b_2) + y_2(a_1b_2 - b_1a_2) \quad \ldots\ldots\ldots(3),\end{aligned}$$

also
$$(A_1B_1Z_1Z_2) = (A_2B_2X_1X_2);$$

$$\therefore \frac{a_2}{b_2} = \frac{z_1 - a_1}{z_1 - b_1} : \frac{z_2 - a_1}{z_2 - b_1};$$

$$\therefore (a_2 - b_2)(z_1z_2 + a_1b_1) = z_1(a_1a_2 - b_1b_2) - z_2(a_1b_2 - b_1a_2).$$

Writing $y_1$ for $-z_1$ and $y_2$ for $-z_2$,

$$(a_2 - b_2)(y_1y_2 + a_1b_1) = -y_1(a_1a_2 - b_1b_2) + y_2(a_1b_2 - b_1a_2) \quad \ldots(4).$$

Square (3) and (4) and subtract

$$\begin{aligned}(a_1 - b_1)^2(y_1y_2 + a_2b_2)^2 - (a_2 - b_2)^2(y_1y_2 + a_1b_1)^2&\\ = 4y_1y_2(a_1a_2 - b_1b_2)(a_1b_2 - b_1a_2)&;\end{aligned}$$

$$\therefore (a_1 - b_1)^2(y_1y_2 - a_2b_2)^2 - (a_2 - b_2)^2(y_1y_2 - a_1b_1)^2 = 0;$$

$$\therefore \frac{(y_1y_2 - a_1b_1)^2}{(a_1 - b_1)^2} = \frac{(y_1y_2 - a_2b_2)^2}{(a_2 - b_2)^2}.$$

Hence the given expression is constant.

**151.** The expression

$$\frac{\{(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)\}^2}{(X_1X_2\,Y_1Y_2)}$$

will be denoted by $H\{{*}A_1{*}B_1X_1X_2Y_1Y_2\}$ or, shortly, by $H\{A_1B_1\}$.

It is thus seen that, if from any point $S$ on a conic two fixed points $A$, $B$ are projected into $A_1$, $B_1$ on a line, on which the conic determines an involution of which $X_1$, $X_2$ and $Y_1$, $Y_2$ are any pair of conjugate points, then $H$ is constant.

It should be noticed that

$$H(ABX_1X_2Y_1Y_2)=H(ABY_1Y_2X_1X_2),$$
$$H(ABX_1X_2Y_1Y_2)=H(BAX_1X_2Y_1Y_2),$$
$$H(ABX_1X_2Y_1Y_2)=H(ABX_2X_1Y_1Y_2),$$
$$H(ABX_1X_2Y_1Y_2)=H(ABX_1X_2Y_2Y_1).$$

Hence the expression consists of three pairs of elements, and

($a$) the elements in any one, or more than one, of the pairs can be interchanged without altering the value of the expression ;

($b$) the elements in the second and third pairs, that is the pairs of conjugate points of the involution, may be similarly interchanged.

If $X_1\equiv X_2\equiv X$ and $Y_1\equiv Y_2\equiv Y$, then

$$H\{AB\}\equiv\left(\frac{1+(ABXY)}{1-(ABXY)}\right)^2=\{2(AXYB)-1\}^2.$$

In this case, if two $H$'s are equal, *either*

$$(AXYB)=(A'X'Y'B') \textit{ or } (AXYB)+(A'X'Y'B')=1,$$

*i.e.* $$(AXYB)=1-(A'X'Y'B')=(B'X'Y'A'),$$

where the dashed letters refer to the second $H$.

Thus if the elements $X_1X_2$, $Y_1Y_2$ are the same in both $H$'s,

either $$(AXYB)=(A'XYB')$$

or $$(AXYB)=(B'XYA').$$

Therefore $$(AXYB)=(B'YXA')$$

or $$(AXYB)=(A'YXB').$$

Thus either $XY$, $AB'$, $BA'$ or $XY$, $AA'$, $BB'$ form an involution.

On reference to the figure it will be seen that if $X_1X_2$, $Y_1Y_2$ are fixed, the ranges described by $A_1$ and $B_1$ are projective. On reference to the analytical expression, viz. $\left\{\frac{y_1y_2-a_1b_1}{(a_1-b_1)\sqrt{y_1y_2}}\right\}^2$, it will be seen that if this expression is constant $B_1$ will describe one or other of two ranges projective with that described by $A_1$ according as the + or − sign is given to the square root of the constant.

If $H\{AB\}$ is given and also either $A$ or $B$, there are two positions of the other according as the + or − sign is taken.

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* *Note.* When dealing with a conic the points which are projected from any point on the conic are frequently substituted for $A_1B_1$ in this expression.

The expression $(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)$ may be denoted by $h\{A_1B_1X_1X_2Y_1Y_2\}$ or shortly by $h\{A_1B_1\}$. It is such that, if $A_1$, $B_1$ be the projections of any two fixed points on a conic from a variable point on the conic upon a fixed line on which the conic determines an involution of which $X_1X_2$, $Y_1Y_2$ are two fixed pairs of conjugates, it is constant, except in regard to sign. If $X_1$, $X_2$, $Y_1$, $Y_2$ are unaltered in order it does not change its sign.

It should be noticed that

(i)
$$h\{ABX_1X_2Y_1Y_2\}=-h\{BAX_1X_2Y_1Y_2\},$$
$$h\{ABY_1Y_2X_1X_2\}=-h\{ABX_1X_2Y_1Y_2\}.$$

Hence, if $A$ and $B$ be interchanged or if the pair of conjugates $X_1X_2$ be interchanged with the pair of conjugates $Y_1Y_2$, the sign of the expression is changed.

(ii)
$$h\{ABX_1X_2Y_1Y_2\}=(X_1X_2Y_1Y_2)\times h\{ABX_1X_2Y_2Y_1\},$$
$$h\{ABX_1X_2Y_1Y_2\}=(X_1X_2Y_1Y_2)\times h\{ABX_2X_1Y_1Y_2\}.$$

(iii) The expression $\dfrac{h\{ABX_1X_2Y_1Y_2\}}{\sqrt{(X_1X_2Y_1Y_2)}}$ is such that it is constant when any other pair of conjugates of the involution are substituted for $X_1X_2$ and for $Y_1Y_2$.

If
$$\frac{h\{ABX_1X_2Y_1Y_2\}}{\sqrt{(X_1X_2Y_1Y_2)}}=\frac{h\{A'B'X_1'X_2'Y_1'Y_2'\}}{\sqrt{(X_1'X_2'Y_1'Y_2')}},$$

and $X$ and $Y$ and $X'$ and $Y'$ are the double points of the involutions, it follows that

$$(AXYB)=(A'X'Y'B').$$

If
$$h\{ABX_1X_2Y_1Y_2\}=h\{A'B'X_1X_2Y_1Y_2\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

after dividing both sides by $\sqrt{(X_1X_2Y_1Y_2)}$ the double points may be substituted for $X_1X_2$ and for $Y_1Y_2$ and then

$$(AXYB)=(A'XYB')=(B'YXA').$$

Therefore $AB'$, $XY$, $BA'$ form an involution.

If in (1) the sign of one expression is changed, it follows that

$AA'$, $XY$, $BB'$ form an involution.

The analytical expression for $h\{A_1B_1X_1X_2Y_1Y_2\}$ is $\dfrac{y_1y_2-a_1b_1}{y_2(b_1-a_1)}$.

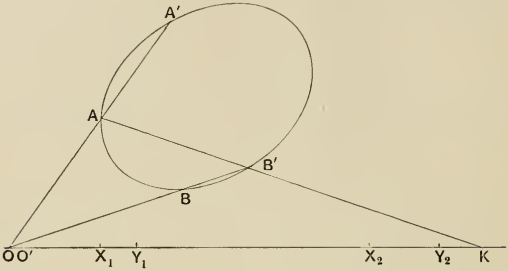
If the elements $X_1$, $X_2$, $Y_1$, $Y_2$ which determine the involution are given and $h\{AB\}$ is constant, $A$ and $B$ describe projective ranges, and if $A$ is given, $B$ is uniquely determined.

If $\dfrac{h\{AB\}}{\sqrt{(X_1X_2Y_1Y_2)}}$ and the involution are given, the ranges $A$ and $B$ are projective, and if $A$ is given, $B$ is uniquely determined.

### 152. Extension of the Involution Property of a Conic.

*If through any point $O$ three chords be drawn the first two of which meet a conic in $AA'$, $BB'$, while the conic determines on the third an involution $X_1X_2$, $Y_1Y_2$, then*

$$h\{ABX_1X_2Y_1Y_2\} = h\{A'B'Y_1Y_2X_1X_2\}.$$



Let $AB'$ meet the base of the involution at $K$.

$h\{ABX_1X_2Y_1Y_2\}$ by projection from $B'$ becomes $(KX_1Y_2O) - (KY_1X_2O)$.

$h\{A'B'Y_1Y_2X_1X_2\}$ by projection from $A$ becomes $(OY_1X_2K) - (OX_1Y_2K)$.

But
$$\begin{aligned}(OY_1X_2K) - (OX_1Y_2K) &= (KX_2Y_1O) - (KY_2X_1O)\\ &= 1 - (KY_1X_2O) - 1 + (KX_1Y_2O)\\ &= (KX_1Y_2O) - (KY_1X_2O).\end{aligned}$$

Therefore
$$h\{ABX_1X_2Y_1Y_2\} = h\{A'B'Y_1Y_2X_1X_2\}.$$

*Deductions:*
$$\frac{h\{ABX_1X_2Y_1Y_2\}}{\sqrt{(X_1X_2Y_1Y_2)}} = \frac{h\{A'B'Y_1Y_2X_1X_2\}}{\sqrt{(Y_1Y_2X_1X_2)}}$$

and
$$H\{ABX_1X_2Y_1Y_2\} = H\{A'B'X_1X_2Y_1Y_2\}.$$

*Converse of extension of the Involution Property.*

*If $A$, $A'$, $B$, $B'$ are points on a conic which determines an involution $X_1X_2$, $Y_1Y_2$ on a given line, then if $h\{ABX_1X_2Y_1Y_2\} = h\{A'B'Y_1Y_2X_1X_2\}$ the three lines $AA'$, $BB'$ and $X_1X_2Y_1Y_2$ are concurrent.*

Let $AA'$ meet $X_1X_2$ in $O$ and let $BB'$ meet $X_1X_2$ in $O'$.

$h\{ABX_1X_2Y_1Y_2\}$ by projection from $B'$ becomes $(KX_1Y_2O') - (KY_1X_2O')$.

$h\{A'B'Y_1Y_2X_1X_2\}$ by projection from $A$ becomes $(OY_1X_2K) - (OX_1Y_2K)$.

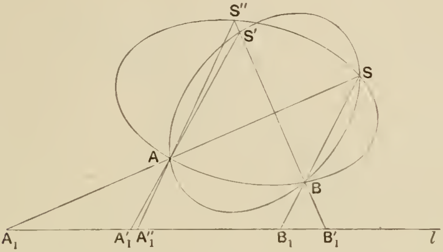
$$\begin{aligned}\therefore\ (KX_1Y_2O') - (KY_1X_2O') &= (OY_1X_2K) - (OX_1Y_2K)\\ &= (KX_1Y_2O) - (KY_1X_2O).\end{aligned}$$

Therefore $h\{KO'X_1X_2Y_1Y_2\}$ and $h\{KOX_1X_2Y_1Y_2\}$ have the same values and $O$ and $O'$ must coincide.

*If $X_1$, $X_2$, $Y_1$, $Y_2$ be any four fixed points on a straight line $l$, $A_1$, $B_1$ and $A_1'$, $B_1'$ any other two pairs of points on this line such that*

$$h\{A_1B_1X_1X_2Y_1Y_2\}=h\{A_1'B_1'X_1X_2Y_1Y_2\};$$

*then, if $S$ and $S'$ be any two points in the plane and $A_1S\,.\,A_1'S'$ be $A$ and $B_1S\,.\,B_1'S'$ be $B$, a conic can be described through $A$, $B$, $S$, $S'$ such that it determines the involution $X_1X_2$, $Y_1Y_2$ on $l$.*



If the conic which determines the given involution on $l$ and passes through $A$, $B$, $S$ does not pass through $S'$, let it meet $B_1'BS'$ in $S''$. Join $S''$ to $A$ to meet $l$ in $A_1''$.

Then $h\{A_1B_1\}=h\{A_1''B_1'\}$. But $h\{A_1B_1\}=h\{A_1'B_1'\}$;

$$\therefore\ h\{A_1''B_1'\}=h\{A_1'B_1'\}.$$

Therefore $A_1'$ and $A_1''$ coincide. This proves the theorem.

By joining $A_1$ to $S'$ and $A_1'$ to $S$ a point $A'$ is obtained in lieu of $A$; and, by joining $B_1$ to $S'$ and $B_1'$ to $S$ a point $B'$ is obtained in lieu of $B$.

The condition given above likewise assures that a conic can be described through $S$, $S'$, $A'$ and $B'$ to determine the given involution on $l$.

### 153. Extension of Carnot's Theorem.

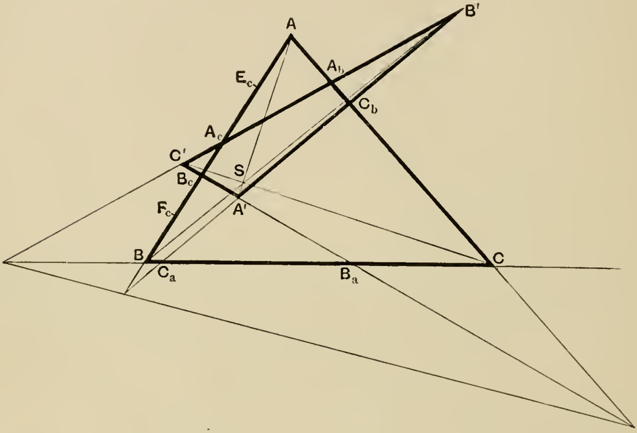
*If $ABC$ be any triangle and the polars of $A$, $B$, $C$ with respect to any conic meet the sides $AB$, $BC$, $CA$ in $A_cB_c$, $C_aB_a$, $A_bC_b$ as in the figure, then*

$$\frac{AA_c\,.\,AB_c}{BA_c\,.\,BB_c}\cdot\frac{BC_a\,.\,BB_a}{CC_a\,.\,CB_a}\cdot\frac{CA_b\,.\,CC_b}{AA_b\,.\,AC_b}=1.$$

Let the polars of $A$, $B$, $C$ meet in $A'$, $B'$, $C'$. Then the triangles $ABC$, $A'B'C'$ are in perspective (Art. 103 ($a$)).

Hence a conic can be described through $A_b$, $A_c$, $B_a$, $B_c$, $C_a$, $C_b$ and the above expression is equal to unity by Carnot's Theorem.

If the conic meets the sides of the triangle $ABC$ in real points $E_a$, $F_a$, $E_b$, $F_b$, $E_c$, $F_c$ the connexion between this result and Carnot's theorem may be shown as follows. Since conics through $E_a$, $F_a$, $E_b$, $F_b$ meet $AB$ in $E_c$, $F_c$ and in $A$, $B$, and $A_c$, $B_c$ are conjugates of the involution determined by $A$, $B$, $E_c$, $F_c$ (Art 59), a conic can be described through $E_a$, $F_a$, $E_b$, $F_b$ and $A_c$ and $B_c$. Hence Carnot's theorem holds for $E_a$, $F_a$, $E_b$, $F_b$ and $A_c$ and $B_c$, and $A_c$ and $B_c$ can be substituted for $E_c$ and $F_c$.



Similarly, the points where the polars meet the other sides may be substituted for the double points of the involutions on these sides.

### 154. Extension of Desargues' Theorem.

*Each conic of a system of conics through four fixed points determines on every line in its plane an involution and the double points if real of the involutions on any line form an involution. If one of the involutions is given by* $X_1X_2$, $Y_1Y_2$, *and the involution of double points by* $A_1B_1'$ *and* $B_1A_1'$, *then these points are connected by the relation*

$$h\,\{A_1B_1X_1X_2Y_1Y_2\} = h\,\{B_1'A_1'Y_1Y_2X_1X_2\}.$$

In the figure of page 345 let $A$, $B$, $S$, $S'$ be the four given points through which the conics are described and let one of the conics determine on the line $l$ an involution of which $X_1X_2$, $Y_1Y_2$ are two pairs of conjugate points.

Then $$h\{A_1B_1X_1X_2Y_1Y_2\}=h\{A_1'B_1'X_1X_2Y_1Y_2\}$$
$$=h\{B_1'A_1'Y_1Y_2X_1X_2\};$$
$$\therefore\ \frac{h\{A_1B_1X_1X_2Y_1Y_2\}}{\sqrt{(X_1X_2Y_1Y_2)}}=\frac{h\{B_1'A_1'Y_1Y_2X_1X_2\}}{\sqrt{(Y_1Y_2X_1X_2)}}.$$

If the involution $X_1X_2$, $Y_1Y_2$ has real double points $X$, $Y$, they may be substituted for $X_1$, $X_2$ and for $Y_1$, $Y_2$ in this expression and
$$(A_1XYB_1)=(B_1'YXA_1').$$

Thus $X$, $Y$ are conjugate points of the involution determined by $A_1B_1'$ and $B_1A_1'$.

Since
$$(A_1X_1Y_2B_1)-(A_1Y_1X_2B_1)=(A_1'X_1Y_2B_1')-(A_1'Y_1X_2B_1')\ \ldots(1),$$
$$\therefore\ (X_1A_1B_1Y_2)-(X_1A_1'B_1'Y_2)=(Y_1A_1B_1X_2)-(Y_1A_1'B_1'X_2).$$

Now $A_1B_1'$ and $B_1A_1'$ may be looked upon as determining an involution. Then the above becomes
$$h\{X_1Y_2,\ A_1B_1',\ A_1'B_1\}=h\{Y_1X_2,\ A_1B_1',\ A_1'B_1\};$$
$$\therefore\ \frac{h\{X_1Y_2,\ A_1B_1',\ A_1'B_1\}}{\sqrt{(A_1B_1'\,A_1'B_1)}}=\frac{h\{Y_1X_2,\ A_1B_1',\ A_1'B_1\}}{\sqrt{(A_1B_1'\,A_1'B_1)}}.$$

Hence for $A_1B_1'$, $A_1'B_1$ may be substituted in this and consequently in (1) any two pairs of conjugate points in the involution of double points. These double points when real are the intersections of conics of the system with the line $l$.

Hence the following is obtained:

*If any three conics are described through four fixed points and the first two meet a given line in $K$ and $K'$, and $L$ and $L'$ and the third determines an involution on the line of which $X_1X_2$, $Y_1Y_2$ are any two pairs of conjugate points, then*
$$h\{KLX_1X_2Y_1Y_2\}=h\{L'K'X_1X_2Y_1Y_2\}.$$

The theorem may also be stated thus:

*If a system of conics be described through four fixed points, they determine on every line in their plane an involution. Let $KK'$, $LL'$ be any two pairs of conjugate points of this involution. Let $X_1X_2$, $Y_1Y_2$ be any two pairs of conjugate points of the involution determined by any one of the conics on the same line. Then*
$$h\{KLX_1X_2Y_1Y_2\}=h\{L'K'X_1X_2Y_1Y_2\},$$
*and* $$h\{X_1Y_2KK'LL'\}=h\{Y_1X_2KK'LL'\}.$$

The above relation between the two involutions is reciprocal. If $KK'$, $LL'$ is termed involution (1), and $X_1X_2$, $Y_1Y_2$ involution (2), it is known that the double points of (2) when real are a pair of conjugate points of (1). Hence the double points of (1) when real are a pair of conjugate points of (2). But (2) may be the involution determined by any conic of the system. Hence the double points of (1) are a pair of conjugate points of the involution determined on the line by any conic of the system.

Hence, *if two conics be described through four points $R$, $S$, $T$, $U$ to touch a given line $l$ at $X$ and $Y$, then $X$ and $Y$ are a pair of conjugate points of the involution determined by $l$ on any conic through $R$, $S$, $T$, $U$.* This is otherwise obvious when the double points of involution (2) are real.

The extension of Desargues' theorem may also be expressed in another form.

It has already been proved (Art. 153) that Carnot's theorem holds when for the points of intersection of a side with the conic are substituted the points where the polars of the vertices, on that side, meet the side in question. Hence the following is obtained.

*If a system of conics be described through four points $C_b$, $A_b$, $B_a$, $C_a$ and any transversal meet the lines $C_bA_b$ and $B_aC_a$ in $A$ and $B$, then the polars of $A$ and $B$ with respect to any conic of the system meet the transversal in a pair of conjugate points of the same involution as that in which the transversal is met by conics of the system which meet it in real points.*

## EXAMPLES.

(1) Show that, if a conic meet the sides $AC$, $CB$ of a triangle in two pairs of points $B_1$, $B_2$, $A_1$, $A_2$ whose ratios are $b_2b_1$ and $a_2a_1$ and determines on the side $AB$ an involution, the ratio of two pairs of conjugates of which are $x_1x_2$, $y_1y_2$, then

$$(x_1+x_2)\{1+y_1y_2a_1b_1a_2b_2\}=(y_1+y_2)\{1+x_1x_2a_1b_1a_2b_2\}.$$

Let $A_1B_1$ and $A_2B_2$ meet $AB$ in $C_3$ and $C_4$. Then we have $h\{AC_4\}=h\{C_3B\}$. Let $c_3$ and $c_4$ be the ratios of $C_3$ and $C_4$. Express in terms of ratios with $A$ and $B$ as origin and afterwards substitute

$$\frac{1}{a_1b_2} \text{ for } c_4 \text{ and } \frac{1}{a_2b_1} \text{ for } c_3.$$

(2) Deduce Carnot's Theorem from (1).

(3) If $X$ and $X'$ be the ratios of the conjugates of $A$ and $B$ in the involution on $AB$ in (1), deduce that

$$a_1a_2b_1b_2XX'=1.$$

(4) Prove that if $A$, $B$ are a pair of conjugate points of the involution on $AB$ in (1) then

$$a_1a_2b_1b_2x_1x_2=-1.$$

(5) Show that $a_1a_2b_1b_2x_1x_2y_1y_2=\dfrac{\overline{x_1+x_2}-\overline{y_1+y_2}}{\overline{\frac{1}{y_1}+\frac{1}{y_2}}-\overline{\frac{1}{x_1}+\frac{1}{x_2}}}$.

(6) Prove that the general value of $h\{ABX_1X_2Y_1Y_2\}$ with any origin is

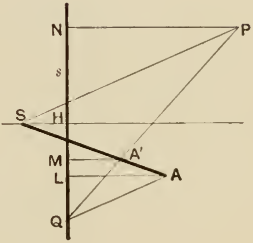
$$\frac{(a+b)(x_1x_2-y_1y_2)+(x_1+x_2)(y_1y_2-ab)-(y_1+y_2)(x_1x_2-ab)}{(b-a)}.$$

# ADDENDUM

## NON-PROJECTIVE PROOFS OF PROPERTIES OF STRAIGHT LINES AND CIRCLES

### The straight line.

1. *Through $A$, one of three fixed collinear points $A$, $A'$, $S$, a variable line $AQ$ is drawn to meet a fixed line $s$ at $Q$. A line through $S$ parallel to $AQ$ meets $QA'$ in $P$. Then the locus of $P$ is a line parallel to $s$.*



Let $L$, $M$, $N$, $H$ be the feet of the perpendiculars from $A$, $A'$, $P$ and $S$ respectively on $s$.

Then $$\frac{PN}{A'M}=\frac{PQ}{A'Q}=\frac{SA}{A'A},$$

since $SP$ and $QA$ are parallel.

Therefore $$PN=A'M\,.\,\frac{SA}{A'A},$$
and is constant.

2. (*a*) *If the three pairs of sides of two triangles are parallel, the lines joining the three pairs of corresponding vertices are concurrent.*

Let $ABC$ and $A'B'C'$ be two triangles whose corresponding sides are parallel. Let $AA'$ meet $BB'$ in $S$ and $CC'$ in $S_1$. It will be shown that $S$ and $S_1$ coincide.

In the similar triangles $SAB$ and $SA'B'$,

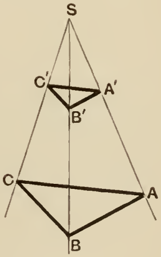
$$\frac{SA'}{SA}=\frac{A'B'}{AB}.$$

Similarly $$\frac{S_1A'}{S_1A}=\frac{C'A'}{CA}.$$

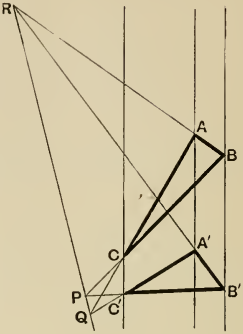
But $$\frac{A'B'}{AB}=\frac{C'A'}{CA}.$$

Therefore $$\frac{SA'}{SA}=\frac{S_1A'}{S_1A},$$

and $S$ and $S_1$ coincide.



(*b*) *If the lines joining three pairs of corresponding vertices of two triangles are parallel, then the points of intersection of corresponding pairs of sides are collinear.*



Let $ABC$ and $A'B'C'$ be the two triangles whose corresponding vertices are on the parallel lines $AA'$, $BB'$, $CC'$.

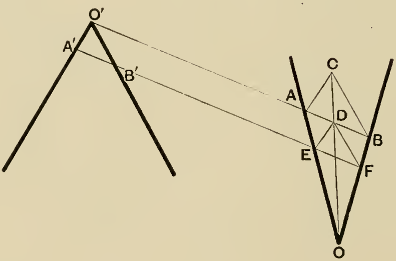
Let $CB . C'B'$, $AC . A'C'$ and $BA . B'A'$ be $P$, $Q$ and $R$.

Then
$$\frac{BP}{CP}=\frac{BB'}{CC'},$$
$$\frac{CQ}{AQ}=\frac{CC'}{AA'},$$
$$\frac{AR}{BR}=\frac{AA'}{BB'}.$$

Therefore
$$\frac{BP . CQ . AR}{CP . AQ . BR}=1,$$

and by Menelaus' theorem $P$, $Q$, $R$ are collinear.

3. *To draw a straight line in a given direction so that the intercepts made on it by the sides of two given angles shall be equal.*



Let the two given angles be $AOB$ and $A'O'B'$. Through $O'$ draw a line in the given direction to meet $OA$ and $OB$ in $A$ and $B$. Through $A$ and $B$ draw lines $AC$ and $BC$, parallel respectively to $O'A'$ and $O'B'$, to meet at $C$. Join $OC$ to meet $AB$ at $D$ and through $D$ draw $DE$ and $DF$, parallel to $CA$ and $CB$, to meet $OA$ and $OB$ at $E$ and $F$. The line $EF$ is the required line.

Let $EF$ meet $O'A'$ and $O'B'$ in $A'$ and $B'$.

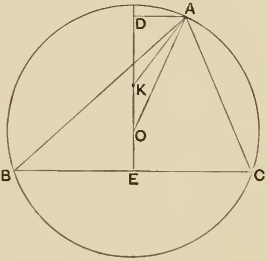
Then, since $\frac{OE}{EA}=\frac{OD}{DC}=\frac{OF}{FB}$, $EF$ is parallel to $AB$. Therefore the figure $O'A'ED$ is a parallelogram and $O'A'$ is equal to $DE$. But the triangles $O'A'B'$ and $DEF$ are similar, and, since $O'A'$ equals $ED$, they are equal. Therefore $A'B'$ equals $EF$ and $EF$, since it is parallel to $AB$, is the required line.

4. *To describe a triangle $ABC$ given* (1) *the base $BC$*, (2) *the difference of base angles $C-B$, and* (3) *the product of sides $AB.AC$.*

Let the given difference of base angles $=\theta$ and let $\frac{bc}{a^2}=\lambda$, where $a$, $b$, $c$ are the sides of the required triangle.

Take any circle centre $O$ and draw $OA$ making an angle $\theta$ with a diameter $OD$.

From $OD$ cut off $OK$ so that $\frac{OA}{OK}=4\lambda$.



Cut off $KE$ equal to $KA$ and draw $BEC$ perpendicular to $DO$. The triangle $ABC$ will be shown to be similar to the required triangle which can therefore be constructed. Draw $AD$ perpendicular to $OK$, then

$$KE^2=KA^2=KD^2+OA^2-OD^2,$$

$$\therefore\ OD^2-KD^2=OC^2-KE^2=OE^2+EC^2-KE^2,$$

$$\therefore\ OK(OD+KD)=EC^2-OK(OE+KE),$$

$$\therefore\ EC^2=2\,OK.DE.$$

Therefore since $AB.AC=2.OA.DE$ and $\frac{OA}{OK}=4\lambda$,

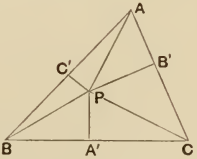
$$\frac{AB.AC}{BC^2}=\frac{2.OA.DE}{4\,EC^2}=\lambda.$$

Also $$C-B=A\hat{O}D=\theta.$$

Therefore the triangle $ABC$ is similar to the one required.

5. *If $A'$, $B'$, $C'$ be three points situated on the sides of the triangle $ABC$, the necessary and sufficient condition that the perpendiculars through $A'$, $B'$, $C'$ to the sides of the triangle should be concurrent is that*

$$BA'^2+CB'^2+AC'^2=AB'^2+BC'^2+CA'^2.$$



Let the perpendiculars through $A'$ and $C'$ meet at $P$. Draw $PB'$ perpendicular to $AC$.

Then $$BP^2=BA'^2+PA'^2,\quad BP^2=BC'^2+PC'^2,$$
$$AP^2=AC'^2+PC'^2,\quad AP^2=AB'^2+PB'^2,$$
$$CP^2=CB'^2+PB'^2,\quad CP^2=CA'^2+PA'^2,$$

Hence $BA'^2+AC'^2+CB'^2=AB'^2+CA'^2+BC'^2$.

Conversely, if the given relation holds, and $B''$ is the foot of the perpendicular from $P$ on $AC$, then

$$CB'^2-AB'^2=CB''^2-AB''^2,$$

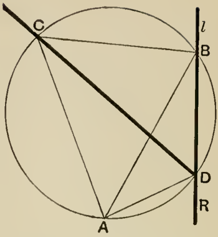
or $$CA\,(CB'+AB')=CA\,(CB''+AB''),$$

$$\therefore\ CB'+AB'=CB''+AB''.$$

Hence $2.B''B'$ is zero and $B'$ and $B''$ must coincide.

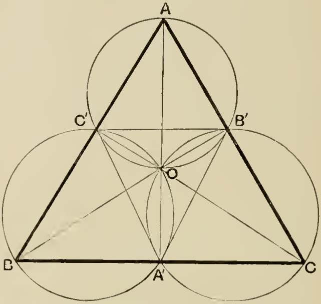
6. *Triangles given in species.*

(*a*) *Given a triangle in species, if one vertex is fixed and another moves along a straight line, the locus of the third vertex is likewise a straight line.*



Let $A$ be the fixed vertex of the triangle $ABC$ and let $B$ move along the straight line $l$. Describe a circle round $ABC$ to meet $l$ at $D$. Since the angle $ADR=ACB$, the point $D$ is fixed. Join $CD$. Since the angle $CDB=BAC$, the line $CD$ is fixed. Hence this line is the locus of $C$.

(*b*) *A triangle given in species, and inscribed in a given triangle, may be divided into three triangles each of which is given in species and has one vertex fixed, while its other two vertices are on sides of the given triangle.*



Let $ABC$ be the given triangle and $A'B'C'$ the inscribed triangle given in species. If circles be described through $C$, $A'$, $B'$ and $B$, $A'$, $C'$ to meet in $O$, the sum of the angles $C'BA'$, $C'OA'$, $A'CB'$ and $A'OB'$ is four right angles and therefore the sum of the angles $C'AB'$ and $C'OB'$ is two right angles. Hence a circle can be described through $A$, $C'$, $O$, $B'$.

The angle

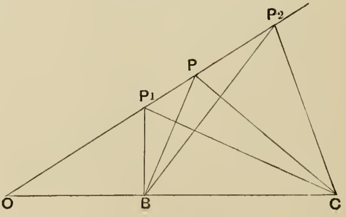
$$\begin{aligned} BOC &= BAC + ABO + ACO \\ &= BAC + C'A'O + OA'B' \\ &= BAC + C'A'B'. \end{aligned}$$

As these angles are given the angle $BOC$ is constant and $O$ is situated on a given circle through $B$ and $C$. Similarly, it is on a given circle through $B$ and $A$ and is therefore a fixed point.

Also the angle $OC'A'$ equals the angle $OBC$ and is therefore constant. Hence the triangle $OC'A'$ is given in species. One of its vertices $O$ is fixed and the other two move along fixed lines $AB$ and $BC$.

7. *Areas.*

(*a*) *If $A_1$ and $A_2$ be the areas of the triangles $P_1BC$ and $P_2BC$ and $P$ be any point on the line $P_1P_2$, then the area $PBC$ equals* $\dfrac{PP_2 . A_1 + P_1P . A_2}{P_1P_2}$.



Let $P_1P_2$ meet $BC$ in $O$, and let $p_1$, $p_2$, $p$ be the perpendiculars from $P_1$, $P_2$, $P$ respectively on $BC$.

Then, if $A$ be the area of $PBC$,

$$\begin{aligned} A_1 : A : A_2 &:: p_1 : p : p_2 \\ &:: OP_1 : OP : OP_2. \end{aligned}$$

Hence it follows that

$$A = \frac{PP_2 . A_1 + P_1P . A_2}{P_1P_2}.$$

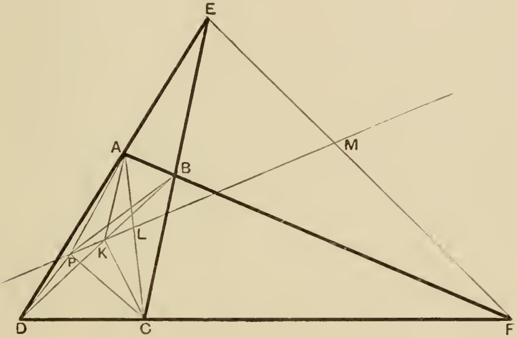
*Conversely, if the above relation holds the point $P$ is on the line joining $P_1$ to $P_2$.*

*Note.* In all cases when an area passes through a zero value its sign must be changed.

(*b*) *If $B_1C_1$, $B_2C_2$, $B_3C_3$, ... and $D_1F_1$, $D_2F_2$, $D_3F_3$, ... be any finite straight lines of limited length in a plane, the locus of a point such that $\Sigma PB_1C_1 = \Sigma PD_1F_1$ is a straight line.*

This result may be easily deduced from (*a*).

8. *At any internal point on the connector of the middle points of the diagonals of a quadrilateral the sum of the areas subtended by one pair of opposite sides is equal to the sum of the areas subtended by the other pair.*



Let $K$, $L$, $M$ be the middle points of the diagonals $BD$, $AC$ and $EF$ of the quadrilateral.

Take $P$ a point on $KML$ inside the quadrilateral.

Then, denoting areas of triangles by their three vertices,

$$PDC + PKC = KDC + PDK,$$

and $$PAB + PKB = ABK + PKA.$$

But $PKC = PKA$, since $L$ is the middle point of $AC$;

and $PKB = PDK$ „ $K$ „ „ „ „ $BD$.

$$\therefore\ PDC + PAB = KDC + KAB.$$

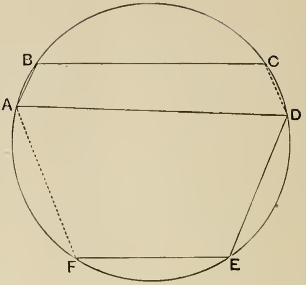
Similarly $$PAD + PBC = KAD + KBC.$$

But $KAB = KAD$ and $KDC = KBC$, since $K$ is the middle point of $BD$.

Therefore $$PDC + PAB = PAD + PBC.$$

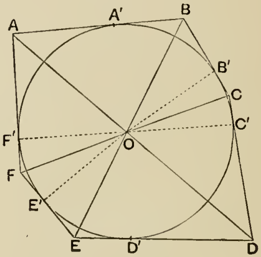
## The Circle.

9. (*a*) *If two pairs of opposite sides of a hexagon inscribed in a circle are parallel, then the remaining pair of opposite sides are also parallel.*



In the hexagon $ABCDEF$, let the pairs of sides $AB$, $ED$ and $BC$, $EF$ be parallel. Join $AD$. Since $AB$ and $BC$ are parallel to $ED$ and $FE$, the angles $ABC$ and $FED$ are equal. But the angles $ADC$ and $ABC$ together equal two right angles as do the angles $FAD$ and $FED$. Therefore $FAD$ and $CDA$ are equal and the lines $CD$ and $AF$ are parallel.

(*b*) *If a hexagon be circumscribed to a circle and two of the lines joining pairs of opposite vertices pass through the centre, then the line joining the third pair of opposite vertices also passes through the centre.*



Let $ABCDEF$ be a circumscribed hexagon, the sides of which touch the circle as in the figure at $A'$, $B'$, $C'$, $D'$, $E'$, $F'$. Let $AD$ and $BE$ pass through the centre $O$. Join $FO$, $CO$.

Then the angle $F'OB'=2\,.\,AOB=2\,.\,EOD$

$$=E'OC'\,;$$

also $$F'OF=E'OF$$

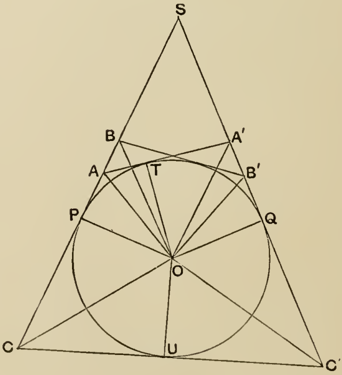
and $$B'OC=COC'.$$

Therefore, since the angles at $O$ are four right angles, the angle

$$FOC=2 \text{ right angles.}$$

Therefore $FO$ and $OC$ are in the same straight line.

10. *If two tangents to a circle intersect two other tangents in* $A$, $A'$ *and* $B$, $B'$ *and* $O$ *be the centre of the circle, then the angles* $AOA'$ *and* $BOB'$ *are equal or supplemental.*



Let $P$, $Q$ and $T$ be the points of contact of the tangents $AS$, $SA'$ and $AA'$. Then the angle

$$AOT=\tfrac{1}{2} \text{ angle } POT$$

and the angle

$$A'OT=\tfrac{1}{2} \text{ angle } TOQ.$$

Therefore the angle $AOA' = \frac{1}{2}$ angle $POQ$
and is constant for different tangents $AA'$, $BB'$, .... Hence

$$AOA' = BOB'.$$

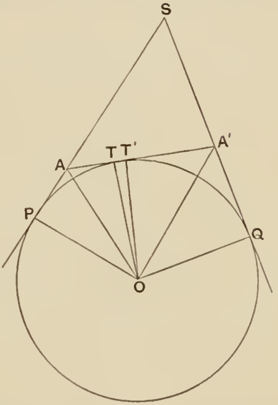
If a tangent $CUC'$ be drawn as in the figure,

$$\text{angle } COC' = \tfrac{1}{2}(\text{angle } POU + \text{angle } UOQ) = \tfrac{1}{2}(2\pi - \text{angle } POQ) = \pi - \tfrac{1}{2}\text{ angle } POQ$$
$$= \pi - \text{angle } AOA'.$$

Therefore angle $COC'$ + angle $AOA' = \pi$.

Conversely :

*If $SP$, $SQ$ be any two tangents to a circle of which $O$ is the centre, and two lines $OA$, $OA'$ containing an angle equal to half $POQ$ be drawn through $O$ to meet $SP$ and $SQ$ in $A$ and $A'$, then $AA'$ is a tangent to the circle.*



Let the tangents from $A$ and $A'$ meet the circle at $T$ and $T'$. Then

$$\text{angle } TOA = \tfrac{1}{2}\text{ angle } POT,$$

and $$\text{angle } T'OA' = \tfrac{1}{2}\text{ angle } T'OQ.$$

$$\therefore\ \text{angle } TOA + \text{angle } T'OA'$$
$$= \tfrac{1}{2}\{\text{angle } POT + \text{angle } T'OQ\}.$$

But

$$\text{angle } TOA + \text{angle } TOT' + \text{angle } T'OA'$$
$$= \tfrac{1}{2}\{\text{angle } POT + \text{angle } TOT' + \text{angle } T'OQ\}.$$

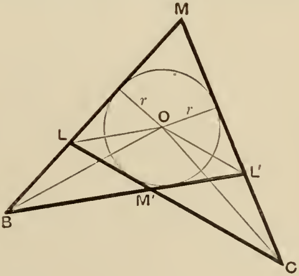
Therefore

$$\text{angle } TOT' = \tfrac{1}{2}\text{ angle } TOT'.$$

Hence the angle $TOT'$ is zero, and $T$ and $T'$ coincide so that $AA'$ is a tangent to the circle.

11. *If $LML'M'$ be a quadrilateral circumscribed to a circle of centre $O$, and $B$ and $C$ are the points of intersection of $ML$, $M'L'$ and of $ML'$, $LM'$ respectively, then*

$$\frac{BL \cdot BL'}{BO^2} = \frac{CL \cdot CL'}{CO^2}.$$



Let $r$ be the radius of the circle. Then the angles $LOC$ and $BOL'$ are equal or supplemental and the angles $LOB$ and $L'OC$ are equal.

Hence considering the areas of the triangles $BOL$ and $COL'$,

$$\frac{BL \cdot r}{OL \cdot OB \sin LOB} = 1$$
$$= \frac{r \cdot L'C}{OL' \cdot OC \sin COL'},$$
$$\therefore\ \frac{BL}{CL'} = \frac{OL \cdot OB}{OL' \cdot OC}.$$

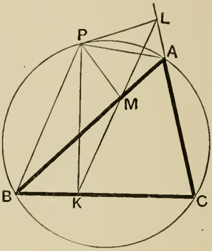
Similarly, from the areas of the triangles $LOC$ and $BOL'$,

$$\frac{BL'}{CL}=\frac{OL'.OB}{OL.OC};$$

$$\therefore\ \frac{BL.BL'}{CL'.CL}=\frac{OB^2}{OC^2}.$$

12. *The feet of the perpendiculars to the sides of a triangle from any point on its circumcircle are collinear.*

Let $ABC$ be the triangle; $P$ any point on its circumcircle and $K$, $L$, $M$ the feet of the perpendiculars from $P$ on the sides of $ABC$. Join $M$ to $K$ and $L$.



Angle $PMK=\pi-$ angle $PBK$

since $P$, $M$, $K$, $B$ are concyclic

$=$ angle $PAC$

since $P$, $A$, $C$, $B$ are concyclic

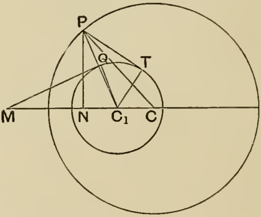
$=\pi-$ angle $PAL$

$=\pi-$ angle $PML$

since $P$, $M$, $A$, $L$ are concyclic.

Therefore $MK$ and $ML$ are in the same straight line.

13. *If $M$ be the point where the radical axis of two circles, whose centres are $C$ and $C_1$, meets their line of centres, then the square of the tangent from any point $P$ on the first circle to the second circle is equal to $2.CC_1.NM$, when $N$ is the foot of the perpendicular from $P$ on the line of centres.*



Let $PT$ be a tangent to the circle centre $C_1$ from $P$ a point on the circle centre $C$, and let $R$ and $r$ be the radii of the circles, then

$$PT^2=C_1P^2-r^2$$

$$=CC_1^2-2.CN.CC_1+R^2-r^2$$

$$=CC_1^2-2.CN.CC_1+CM^2-C_1M^2$$ since the tangents from $M$ to the circles are equal

$$=CC_1^2-2.CN.CC_1+CC_1(CM+C_1M)$$

$$=2.CC_1.CM-2.CC_1.CN$$

$$=2.CC_1.NM.$$

14. *If a chord cut two circles in pairs of points which are harmonic conjugates, the locus of the feet of the perpendiculars from the centres of the circles on the chord is a circle.*

Let $AA'BB'$ be a chord which is cut harmonically by the two circles whose centres are $C$ and $C'$ and radii $R$ and $R'$. Draw the perpendiculars $CN$ and $C'N'$ to $AA'BB'$ and let $NT$ be the tangent from $N$ to the circle whose centre is $C'$. Join $C'N$ and $C'T$.

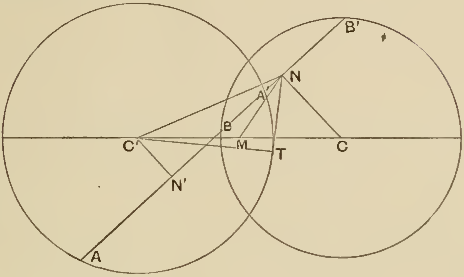
Since $AA'BB'$ is harmonic,

$$NB^2 = NA \, . \, NA' = NT^2 = C'N^2 - R'^2.$$

Also $$NB^2 = R^2 - CN^2.$$

Therefore $$R^2 + R'^2 = CN^2 + C'N^2 = 2C'M^2 + 2MN^2,$$

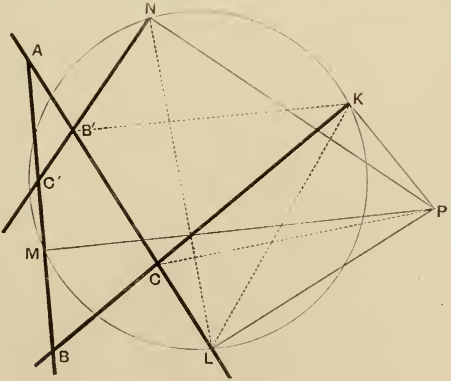
where $M$ is the middle point of $CC'$.



Therefore $MN^2 = \dfrac{R^2 + R'^2}{2} - C'M^2 =$ a constant.

Hence the locus of $N$ is a circle whose centre is at $M$.

15. *If two vertices of a quadrilateral subtend at a given point an angle equal to the angle which the two opposite vertices subtend at the same point, the feet of the perpendiculars from the given point on the sides of the quadrilateral are concyclic.*



Let three of the sides of the quadrilateral be lines $BC$, $CA$, $AB$ and let the fourth side meet these lines in $A'$, $B'$, $C'$ respectively. Let the feet of the perpendiculars from any point $P$ on $BC$, $CA$, $AB$ and $B'C'$ be $K$, $L$, $M$ and $N$ respectively.

Then the angle $NLK = NLP - PLK = NB'P - PCK$

and the angle $NMK = NMP - PMK = NC'P - PBK.$

Therefore $NLK - NMK = (NB'P - NC'P) - (PCK - PBK)$

$$= B'PC' - CPB.$$

Hence, if the angles $B'PC'$ and $CPB$ are equal, the angles $NLK$ and $NMK$ are also equal and the points $K$, $L$, $M$, $N$ are concyclic.

## EXAMPLES.

(1) If $XX'$, $YY'$, $ZZ'$ be the diagonals of a quadrilateral circumscribing a circle, $L$, $M$, $N$ the middle points, and $O$ the centre of the circle, then

$$\frac{OX.OX'}{OL} = \frac{OY.OY'}{OM} = \frac{OZ.OZ'}{ON}.$$

(2) If from the vertices of a triangle perpendiculars be drawn to any straight line, and from the feet of these perpendiculars three perpendiculars are drawn on the opposite sides of the triangle, prove that these perpendiculars are concurrent.

Use Addendum 5.

(3) If lines be drawn from a point $P$ on the circumcircle of a triangle, each making an angle $\alpha$ with the perpendiculars from $P$ on the sides of the triangle, the three points in which they meet the sides of the triangle are collinear.

(4) If $ABC$ be a triangle of which $O$ is the orthocentre, and through $O$ any two straight lines are drawn at right angles to meet the sides $CA$ and $BA$ in $E$ and $F$, and through $B$ and $C$ lines $BN$ and $CN$ are drawn parallel to $OE$ and $OF$ respectively, then $F$, $N$ and $E$ are collinear.

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